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FLOW ANALYSIS THROUGH NON-HOMOGENEOUS SOILS BY COMPUTER

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "FLOW ANALYSIS THROUGH NON-HOMOGENEOUS SOILS BY COMPUTER" submitted by GEOFFREY REGINALD TOMLIN in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis describes the development of a computer program for analysing the flow of water through soils which consist of a number of zones having different permeability; the method of analysis provides for each zone being anisotropic to a different degree, and for the directions of principal permeability being different in each zone.

The analysis involves finding a numerically approximate solution of the basic equation of flow at every node in a mesh superimposed on the flow region. To avoid difficulties which occur in the conventional iteration procedure at zone boundaries, the mesh has three axes; each axis is parallel to the sides of an irregular triangle, drawn so that it coincides with zone boundaries.

The finite difference form of the basic flow equation in terms of three axes in a plane is derived from first principles. The assembly of the computer program is fully described, and the results of several problems for which the program has been used are presented. As presently developed, the program applies to steady-state, two-dimensional, confined flow, but it is suggested that it could be methodically extended to cover cases outside these restrictions.

The writer submits that the method enables analyses to be made of flow regions which are usually regarded as too complicated for solution by existing methods, unless simplifying approximations are made.

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GLOSSARY OF SYMBOLS

A list of the symbols used in the computer program is given on pages D2 to D4.

C_u, C_v, C_w	Coefficients to be used in the finite difference equations. A second subscript signifies the number of the triangle with which the coefficient is associated.
f	Superscript indicating that a value is ficticious, i.e. occurs at an image point.
F	The ratio of a side of a triangle to the sine of an opposite angle.
h	Total head of water.
h_o	Total head of water at a node under consideration.
h'_u, h''_u, h'_v, h''_v, h'_w, h''_w	Potential heads at adjacent nodes to the node under consideration in positive and negative directions along the Ou, Ov and Ow axes. A second subscript signifies the number of the triangle in which the node is situated.
$(h)_u$	The partial derivative of h with respect to u.
$(h)_{uv}$	The second partial derivative of h with respect to u and v.
k	Permeability.
M, N, P	Symbols substituted for u, v, w, and v, w, u, and w, u, v, respectively.
q	Flow through an elemental arc of a circle radius Δs.
Q_1	Outward flow across the curved portion of the perimeter of a semi-circle radius Δs.

Q_2	Inward flow across the diameter of a semi-circle radius Δs.
R	Residual at a node.
s	Any general direction from the centre of an elemental circle.
u, v, w	Symbols signifying each of the three axes in a triangular mesh.
v	Superficial velocity.
x, y	Directions of principal permeability in an anisotropic soil.
x_1, y_1, z_1	Three mutually perpendicular directions in an isotropic soil.
Δs	Radius of an elemental circle.
$\Delta_1 u$, $\Delta_1 v$, $\Delta_1 w$	Node spacings on the three axes of a triangular mesh.
α	Angle between the direction of minimum permeability and any general direction, s.
α_a	Angle between the horizontal and the direction of minimum permeability.
α_u, α_v, α_w	Angles between the direction of minimum permeability and the Ou, Ov and Ow axes.
α'_u, α'_v, α'_w	Angles between the horizontal and the Ou, Ov and Ow axes.

CHAPTER I

INTRODUCTION

1.1 General

The flow of water through a soil may be analysed by one of several methods. The method chosen is governed largely by the accuracy required. The hydraulic conditions set up when water flows through soil under ideal circumstances can be described concisely by simple equations; consequently flow analyses are amenable to mathematical treatment. However, estimates of permeabilities of different soils at a site are frequently only crude and thus an exact mathematical analysis is seldom warranted; also, the boundary conditions of most practical problems are sufficiently complex to make the analysis very difficult by classical methods of mathematics.

The engineer is then forced into making decisions such as: he will accept a less rigorous method of analysis in the light of the uncertainty in the estimates of permeability and the boundary conditions of the soil; he will treat a certain soil as infinitely permeable compared with another soil at the same site if the ratios of permeability of the two are more than, say, ten to one; he will treat a soil as if it were isotropic even though it may be anisotropic. In many instances, the effect of these decisions on the final result will be sufficiently small to be negligible, especially if the quantity of seepage is all that is required. If estimates of critical hydraulic gradients or distribution of pore pressures are required, more careful thought

may be taken in choosing a method of analysis and in making any simplifying assumptions.

To reach a practical solution, the engineer almost always has to estimate values for some missing soil data, and modify the data that is available, so that the final answer is a mixture of intuition and analysis. The writer has endeavoured to devise a method of solution which reduces the intuition factor, and yet is still practical.

1.2 Statement of the problem

In view of the inroads that the digital computer is making on engineering science at the present time, the writer became inclined to investigate its use in regard to seepage problems.

A method of analysis is available which is adaptable to the computer (Scott, 1963, page 133); this method involves a repetitive process of calculations and the end result is approached through successively better approximations. If the boundary conditions of the flow region are simple (i.e., in the extreme, if the flow region is rectangular) and if the soil is homogeneous and isotropic, the length of the computer program using this method is small. However, for such a case, the result could be achieved quite easily by hand, using the same method; alternatively another method of analysis could be used to obtain an equally good result. If the flow region consists of a number of soils of different permeabilities, as may be found in a zoned earth dam or a natural soil deposit with several layers of different materials, the method is much more complex and the assembling of the appropriate computer program is time-consuming. Moreover, if another flow region of similar complexity is

then considered, the time spent in altering the program to fit the conditions of the second case may make it worthwhile to assemble an entirely new program. The approach to the problem using other methods would be extremely difficult or impossible, unless simplifying assumptions were made.

The problem that confronted the writer, and to which this thesis is devoted, was to assemble a computer program which would be applicable to seepage through a non-homogeneous,[1] anisotropic soil section of any desired configuration and complexity. The program could then be used for all types of flow regions, without any alterations.

1.3 Extent and limitations of work carried out

The problem was approached by modifying the conventional procedures involved in the repetitive process of calculations so that cumbersome steps caused by the presence of irregular boundaries would be eliminated. In this way, the number and configuration of different zones of soil was a factor of minor importance in the programming, instead of, as hitherto, a major difficulty. The conventional procedure uses the basic flow equation in two dimensions; this equation is used herein but in a rather unusual form. One outcome of this modification is that, if the solution is attempted by hand, even for the simple case of the rectangular flow region mentioned in SECTION 1.2, the calculations become very laborious. Laborious calculations present no problems in programming and computer work, but cumbersome operations are to be avoided if possible.

[1] A soil which consists of two or more zones, homogeneous in themselves, but having different permeabilities from each other, is regarded as non-homogeneous, and this meaning is used throughout this thesis.

The computer program was assembled so that the total heads at reasonably closely spaced points within the flow region would appear as output, together with coordinates to locate the positions of the points. The program applies to two-dimensional steady-state flow through a non-homogeneous, anisotropic soil for which:

(a) Darcy's law and continuity conditions apply and thus the flow may be described by the basic equation:

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} = 0 ,$$

where x and y denote the directions of principal permeabilities in an anisotropic soil,
$k_{x,y}$ are the coefficients of permeability in the x- and y-directions respectively,
h is the total head of water at an arbitary point within the mass,[1]

(b) inside the boundaries of any zone, the soil is homogeneous, but may be anisotropic,

(c) the boundaries of each zone are defined,

(d) for an anisotropic soil, the principal coefficients of permeability and the direction in which they act are defined; for an isotropic soil, the coefficient of permeability is defined,

(e) none of the flow region boundaries are controlled by gravity, i.e., there is no phreatic line present,

(f) the flow region boundaries are finite in length and are defined.

Item (a) above always applies in practical seepage problems. Darcy's law is known to hold if the seepage velocity of the water is such that the Reynold's number is less than about one[2]; continuity conditions are upheld if

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[1] All symbols have been listed in the Glossary of Symbols.

[2] The Reynold's number in this context is based on superficial velocity and a diameter which is that of a sphere having a volume equal to the quotient of the volume of solids of the sample and the number of grains in the sample.

the soil is saturated within the flow region, since water and soil can be considered incompressible for practical cases. In order to satisfy item (b) for cases in which the permeability varies gradually from point to point within a zone, the zone should be sub-divided into a number of smaller zones. Items (c) and (d) are frequently incomplete as a result of a site investigation alone. The engineer will usually have to make up the deficiencies with estimated data obtained by informed guesswork; he may find it profitable to run the program through the computer several times for the same problem, each time inserting different values for the guessed data; by comparing the predicted flow patterns for each set of data, the worst conditions may be determined. Item (e) is a limitation caused by lack of time available to the writer. The main effect of this limitation is that the program, as it stands, cannot be used for analysing the flow of water through earth dams. The writer believes that the program could be extended, without much difficulty, so that flow problems containing a phreatic line may be solved; such an extension need not involve any loss of generality. The effect of item (f) is that the flow region must be enclosed within a finite area. In a strict sense, such is not the case when the problem involves flow under a spillway situated on an infinite layer of permeable material overlying a horizontal impermeable base. To analyse such a problem, vertical impermeable boundaries would have to be inserted through the permeable layer at some finite distance from the spillway; if the distance were made sufficiently large, the effect of the presence of these boundaries would be negligible.

Limitations to the size of the data for any flow region are listed below. These limitations were fixed by the number of memory storage locations

available in the computer used by the writer. For most practical problems, these limitations will have no effect, but should a problem be encountered where the size of the data is greater than that specified, the program may be modified so that the computer calls upon magnetic tape for storing the additional information. The limitations are that:

(a) the maximum number of zones is 20,

(b) considering the shape of any zone to be a polygon, the maximum number of boundaries of any zone is 20,

(c) the maximum number of impermeable boundaries of the flow region is 30,

(d) the maximum number of boundaries across which water enters or leaves the flow region (i.e., constant total head boundaries) is 30.

In the use of the program, any curved boundaries which exist in the flow region must be approximated by a series of straight lines. Since the data defining zone boundaries is presented to the computer by simply listing coordinates locating the corner points of the polygons, it does not matter how irregular the polygons are. The ratios of principal permeabilities for the soils in each zone may be different, and the direction of stratification in an anisotropic soil in any zone may be different from that in any other zone. These conditions may hardly ever be realized in practice, but the extra labour involved in programming for them is minimal and if they were not included in the program, some generality would be lost.

A review of present methods has been undertaken; the method of analysis developed by the writer is described herein, together with the computer program. Some examples for which the program has been used are presented.

CHAPTER II

REVIEW OF EXISTING METHODS OF FLOW ANALYSIS

2.1 Basis of flow analyses

Methods for analysing seepage problems are based on the condition that, when water flows through a homogeneous, isotropic soil mass, the Laplace equation, set out as follows, is satisfied:

$$\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial y_1^2} + \frac{\partial^2 h}{\partial z_1^2} = 0 , \qquad (2.1a)$$

where h is the total head of water an an arbitary point within the mass, and x_1, y_1, z_1 are three mutually perpendicular directions.

The derivation of equation (2.1a) depends on the assumption that Darcy's law and continuity conditions are upheld. In many practical cases in which the length of the flow region in a direction transverse to the flow is long compared with the width or depth, the flow conditions may be considered as two-dimensional. Equation (2.1a) for such cases becomes modified to:

$$\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial y_1^2} = 0 , \qquad (2.1b)$$

where x_1 and y_1 are perpendicular directions in the plane parallel to flow.

In an isotropic soil through which water is percolating, flow occurs at any point in the direction of the maximum hydraulic gradient at that point. After obtaining the solution to equation (2.1b), curves may be plotted joining points at which the total heads are the same; the maximum hydraulic gradient

at any point acts in a direction normal to the curve of constant head passing through that point. Hence curves which represent the path of an individual water particle may be superimposed on several curves of constant head by ensuring that they intersect each other at right angles. The curves of constant head and those representing the flow paths are called, respectively, equipotential lines and flow lines; the combined representation of equipotential lines and flow lines is called a flow net.

2.2 Methods for analysing flow

The available methods for solving flow problems may be broadly divided into three catagories:

(a) Theoretical methods

These methods entail solving, either directly or by numerical approximation, differential equations representing the equipotential lines and flow lines. The direct method uses the theory of complex variables to study the orthogonality of the two sets of curves. Briefly, by choosing an appropriate equation, the geometry of the flow region can be transformed into another configuration, or map, the flow net in which is known or is obvious. Each point in the flow region before transformation is represented by one point in the mapped region, and, therefore, having ascertained the flow conditions at any point in the mapped region, these conditions can be related to the corresponding point in the original flow region. The application of the complex variable to flow problems has been extensively studied by Russian theoreticians, particularly Polubarinova-Kochina (1962). Harr (1962) gives an explanation of the principles involved.

In the analysis using the methods of numerical approximation, each of the second derivatives in the Laplace equation (equation 2.1b) is expressed in finite difference form. The total head h at any intersection point, or node, on an imaginary rectangular mesh of small, but finite, size is then related to the head at the adjacent nodes. An initial guess is made of the value of the head at each node and, by using the finite difference equation, a better approximation to the correct value can be made. The process is repeated until the difference in values obtained by two successive approximations for all nodes is less than some tolerable value. The method, slightly modified, is sometimes called 'relaxation', in which an assessment is made of the amount by which the guessed head at any node is out-of-balance (in terms of the finite difference equation) with the heads at adjacent nodes. The guessed head is then corrected by that amount, or relaxed; this correction throws the adjacent nodes out-of-balance, so then they are relaxed, and the process repeated until the amount of unbalance is tolerably small. Numerical approximation and relaxation methods were developed largely by R. V. Southwell (see, for example, Shaw and Southwell, 1941, and Southwell, 1946).

(b) Experimental methods

Perhaps the method most widely used by engineers for the solution of flow problems is the graphical construction of the flow net. This method was devised by P. Forchheimer in the early part of the twentieth century, and consists of sketching the flow net, starting with the known boundary conditions, proceeding by trial-and-error, and recognizing the fact that equipotential lines and flow lines are orthogonal, so that the final picture consists of a system of 'curvilinear squares'. This method is described in detail by A. Casagrande

(1937). Typical flow-nets which are useful as a guide have been presented by Gregg (1940).

Hydraulic models may be used to determine the flow conditions in or around an earth structure. The structure is constructed to a suitable scale between two parallel glass plates so that the flow occurs parallel to the plates. A head of water is applied to the upstream side of the structure and when flow has reached a steady-state condition, continuous streams of a suitable dye are introduced at various points on the upstream face, against the side of the glass. The traces of the dye follow flow lines, the significant coordinates of which may be recorded or plotted. Piezometers, in the form of small bore tubes, may be inserted into the flow region at specific points and the total head at those points determined from readings on manometers connected to each piezometer. Knowing the flow-lines and the head at some points, the equipotential lines may by sketched in. Taylor (1948, FIGURE 9.15) shows an example of a model dam. Zangar (1953, FIGURE 38) shows photographs of flow-lines under a model weir and under a cut-off wall.

(c) Analogue methods

The Laplace equation, as well as applying to seepage problems, represents the condition governing potential distribution in several other physical phenomena. The flow of electricity or heat through a conducting medium are examples. The same form of equation describes the displacement of a loaded membrane perpendicular to its plane; also, the capillary flow of a liquid between two closely spaced parallel plates satisfies the condition expressed in equation (2.1b). Some consideration and effort has, in the past, been given to the use of these phenomena for solving seepage problems by analogy.

When electricity passes through a two-dimensional medium with resistivities $R_{x,y}$ in the x- and y- directions respectively, the following law is obeyed:

$$\frac{1}{R_x} \cdot \frac{\partial^2 V}{\partial x^2} + \frac{1}{R_y} \cdot \frac{\partial^2 V}{\partial y^2} = 0 ,$$

where V is the voltage at an arbitary point in the plane. In the analogue, it is necessary to obtain an electrical conductor having directional resistivities inversely proportional to the corresponding directional permeabilities. If a rectangular mesh with sides parallel to the directions of principal permeabilities is imagined to be superimposed on the prototype, the nodes of the corresponding mesh in the analogue may be formed by the intersection of electrical resistors related to the permeabilities. Alternatively, if the soil prototype is isotropic, a suitable electrically conducting paper may be used in the analogue. The model is made geometrically similar to the prototype, and uniform voltages are applied along each of the boundaries representing faces of constant total head. The voltage at any point in the paper analogue or any node in the mesh analogue, is then measured by use of a probe and expressed as a proportion of the total voltage difference. In this way, a picture of the potential distribution is built up throughout the flow region. Cases for which the electric analogue have been used are shown by Zangar (1953).

Although the laws governing hydraulic and thermal flows are analogous, solutions to hydraulic flow problems by analogy with thermal models are not attempted because of difficulties in setting up the apparatus and in measuring the temperatures during the conduct of the test. Usually, the solution of a heat flow problem poses as large a question as the solution of a hydraulic

flow problem.

If some part of a uniformly stretched membrane is displaced a small distance perpendicular to its original plane of location, the displacement, z, of any point from that plane is given by:

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial y_1^2} = 0 ,$$

where x_1 and y_1 are perpendicular directions in the original plane of location.

A membrane having the same shape as the soil prototype is uniformly stretched and displaced at the edges by amounts proportional to the total heads acting at the corresponding edges in the prototype. The displacement of any point on the membrane will then be proportional to the head at the corresponding point in the prototype. The method is limited to cases of homogeneous soils, and, so far as the writer is aware, it has been little used for seepage problems. For a fuller description of the method, reference should be made to Walker (1949), who used a rubber membrane analogue to investigate the motion of charged particles in an electric field.

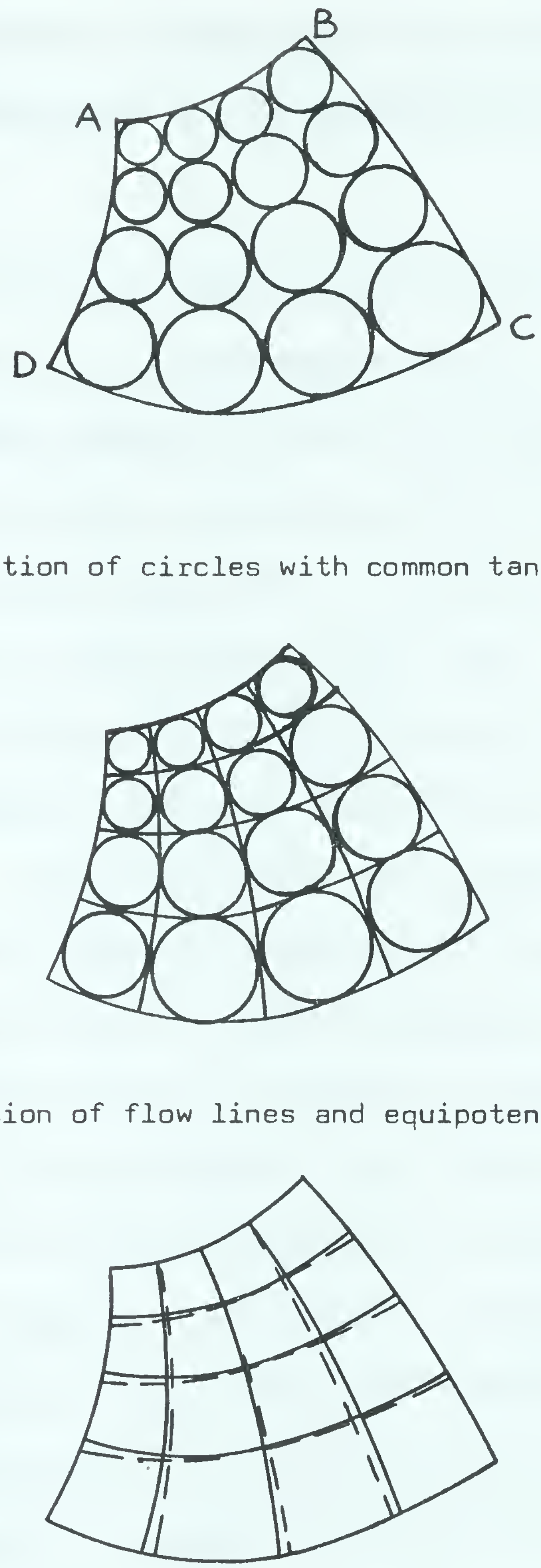
In the capillary flow analogue a viscous fluid is allowed to pass between two closely spaced parallel plates which have the same shape and boundary conditions as the soil profile. The spacing of the plates is determined in conjunction with the viscosity of the fluid so that the Reynold's number is subcritical and laminar flow exists. Polubarinova-Kochina (1962, pp.465-477) describes the method and shows how non-homogeneous soils may be simulated in the analogue.

2.3 Appraisal of existing methods

Because seepage problems occupy the interests of several groups of

persons, such as soil engineers, petroleum engineers, mathematicians and physicists, there exists an abundance of material in the engineering literature dealing with methods of analysis. A few references are made to specific texts in this section; for further material, the bibliographies in the works of Scheidegger (1957) and Harr (1962) should be consulted.

The method of analysis involving the graphical construction of flow nets has several advantages; no equipment or specialised mathematical knowledge is required, it is simple in operation, and as the user progresses through each trial-and-error stage of the operation, he develops an intuitive understanding of the conditions which are most likely to affect the flow net. The method has received severe criticism from Leliavsky (1955), who believes that the basic criterion of forming 'curvilinear squares' was not rigid enough; he suggests that better 'squares' could be formed by constructing an array of circles inside the flow region so that adjacent circles have common tangent points (as shown for the flow region ABCD in FIGURE 2.1). However, since a poorly drawn flow net usually produces no more than a 10 per cent error in the calculation of the seepage quantity, Leliavsky's criticism hardly seems warranted. Two main difficulties are experienced if the flow region consists of a non-homogeneous soil; first, although a relationship exists for the change of direction of flow lines at their intersection with a common boundary between two zones, the expression is awkward to use in the graphical method; secondly, if the zones have different degrees of anisotropy the standard scale transformation (Taylor, 1948, chapter 9) will distort some zones more than others and corresponding points on zone boundaries might not remain coincident. Barron (1948) has presented a technique which reduces these



(a) Construction of circles with common tangent points.

(b) Insertion of flow lines and equipotential lines.

(c) Comparison with constructed flow net (full lines) with sketched flow net (broken lines).

FIGURE 2.1. Circle method of flow net construction (after Leliavsky, 1955)

difficulties, but the quantity of work required for a solution of the case of a non-homogeneous, anisotropic soil by graphical construction still remains large.

Hydraulic models are an ideal means by which the flow net in the prototype can be visualized. For flow regions having a phreatic line as one boundary, the use of models enables this line to be determined directly; however, as Taylor (1948) points out, capillarity effects may detract from the accuracy of the determination of this line. In order to reduce the capillarity effects in the model, a coarse sand is often used. Even so, as Kerr (1959) concludes from some earth dam model tests using sand, the effect of the capillary fringe may not be negligible. As well as the difficulty in locating the capillary fringe accurately, capillarity effects have a further manifestation; there exists a variation in degree of saturation of the soil within the fringe. Because the permeability to water of a soil is dependent on its degree of saturation, permeability should strictly be treated as a variable in flow analyses. Liakopoulos (1964) has conducted tests on a soil inside a tube open at the ends and through which the flow was one-dimensional; from these tests, the relationships between degree of saturation, permeability, and position inside the capillary fringe were determined. The relationships appertain to any seepage condition satisfying Darcy's law, but the difficulty lies in applying them to two-dimensional flow conditions. Because of the doubt which exists on the true effect of capillarity, it is usual to assume that the phreatic line lies along the upper limit of saturation.

The parallel plate analogue permits direct determination of the phreatic line; similation of the capillary fringe in the prototype may be made in the

analogue by correct spacing of the plates, as described by Polubarinova-Kochina (1962). Disadvantages of the parallel plate analogue are that measurement of the potential head at any point is practically unobtainable, and that if the plates are not perfectly parallel, large errors will result.

The electric analogue method has more common use than other methods, since it has several advantages; as well as being simple and convenient, steady-state flow occurs immediately, so that, once the equipment is set up, the procedure is usually completed quickly. One disadvantage is that phreatic lines must be located by trial-and-error. The existence of the capillary fringe has been taken into account in tests using the electric analogue by Vreedenburgh (1936) and Chapman (1960).

The main problem with the complex variable method of flow analysis, as Harr and Deen (1961) point out, lies in finding an equation which will perform the desired transformation. Any polygonal region in one plane may be mapped on to the upper half of an auxiliary plane by means of the Schwarz-Christoffel transformation (Milne-Thomson, 1960). Although the transformation then breaks down into a step-by-step procedure, it might be complicated for any case other than the simplest. Because of this difficulty, the method is of limited value to soil engineers, even though published solutions by this method to certain practical cases (Polubarinova-Kochina, 1962) and lists of conformal transformations (for example, Kober, 1952) are available. For cases in which the boundary conditions are simply described, the method is a mathematical exercise of reasonable length. A classical example is the case attempted by Pavlovsky (1936). He develops the solution to flow under an impervious structure founded on the surface of a homogeneous permeable soil of infinite depth. From his solution, an expression for the distribution of uplift pressure

across the base of the structure can immediately be derived. Similarly, the complex variable theory may advantageously be applied in the case of flow around sheet-piling penetrating some distance into a homogeneous permeable soil of infinite depth, to determine an expression for the upward hydraulic gradient near the exit. It might be argued that these conditions are idealistic but they form a good basis for making a prediction of some of the conditions in more realistic problems.

At the present stage of engineering science, it would be an ill-advised procedure to attempt a solution to the case of flow through a non-homogeneous anisotropic soil by use of complex variable theory. Furthermore, as already explained, such a problem would be cumbersome to solve by the graphical method and difficulties of a practical nature would arise if a model or analogue were used. It is possible to solve the problem using the conventional methods of numerical approximation but, even so, difficulties are encountered. These difficulties are explained in the next section, together with the writer's proposal for overcoming them.

2.4 Numerical analysis of flow problems

For the sake of brevity, the following description of numerical approximate methods considers flow in two dimensions only, through an isotropic soil. The order of complexity would not be enlarged if the third dimension and anisotropy were included.

The Laplace equation (equation 2.1b) can be stated in the following finite difference form:

$$h_1 + h_2 + h_3 + h_4 - 4h_0 = 0, \qquad (2.2a)$$

where h_0 is the total head at a node in an imaginary square mesh, with small sides, superimposed on the flow region,
and $h_{1,2,3,4}$ are the total heads at each of the four adjacent nodes.

The derivation of equation (2.2a) may be found in most textbooks on relaxation methods (for example, Shaw, 1953, chapter 2). Pictorial representation of the mesh is shown in FIGURE 2.2(a). By ensuring that equation (2.2a) is satisfied at all nodes in the mesh, an approximation is obtained to the distribution of total head within the flow region. Generally, the boundaries of the flow region and of the individual zones do not lie along a mesh side (as shown in FIGURE 2.2(b)), and either an approximate boundary may be drawn, following the true boundary as closely as possible but keeping strictly to mesh lines, or a modification to equation (2.2a) is required for those nodes adjacent to the boundary. The modified finite difference equation for nodes near boundaries is (Shaw, 1953, chapter 5):

$$Ah_1 + Bh_2 + Ch_3 + Dh_4 - (E+F)h_o = 0, \qquad (2.2b)$$

where

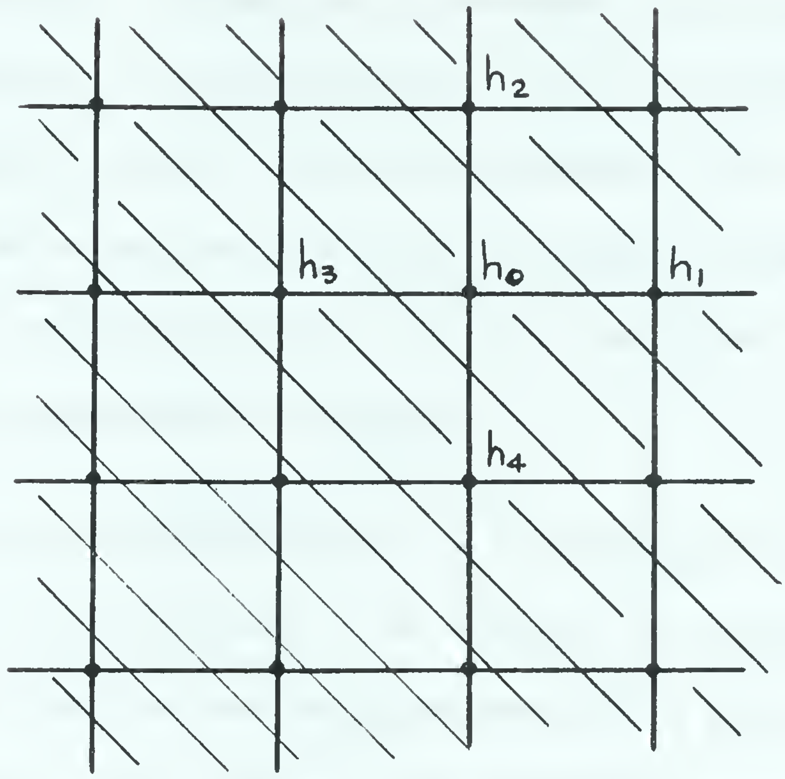
$$A = \frac{2}{a_1(1+a_1)}, \quad C = \frac{2}{(1+a_1)}, \quad E = \frac{2}{a_1},$$

$$B = \frac{2}{a_2(1+a_2)}, \quad D = \frac{2}{(1+a_2)}, \quad F = \frac{2}{a_2},$$

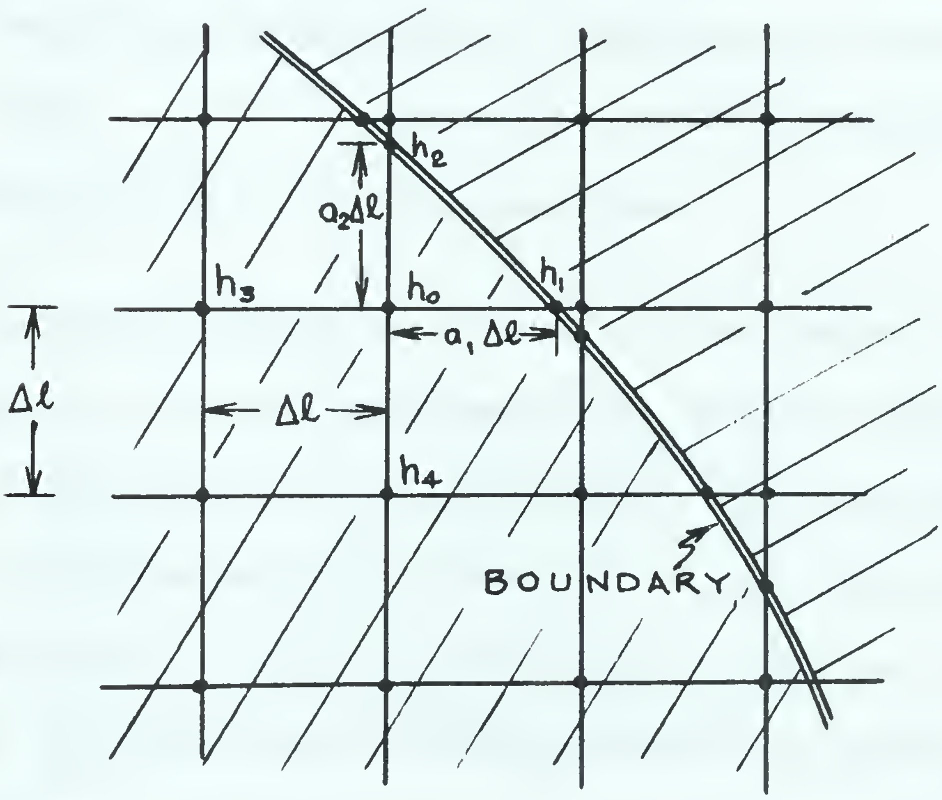
a_1, a_2 are the lengths shown in FIGURE 2.2(b),
and the locations of h_1, h_2, h_3 and h_4 are as shown in FIGURE 2.2(b).

It is to be noted that for each node less than one mesh length from the boundary, the values of the coefficients A,....,F in equation (2.2b) are generally different and depend on the horizontal and vertical distances from the node to the boundary.

The techniques used in numerical approximation have already been



(a)



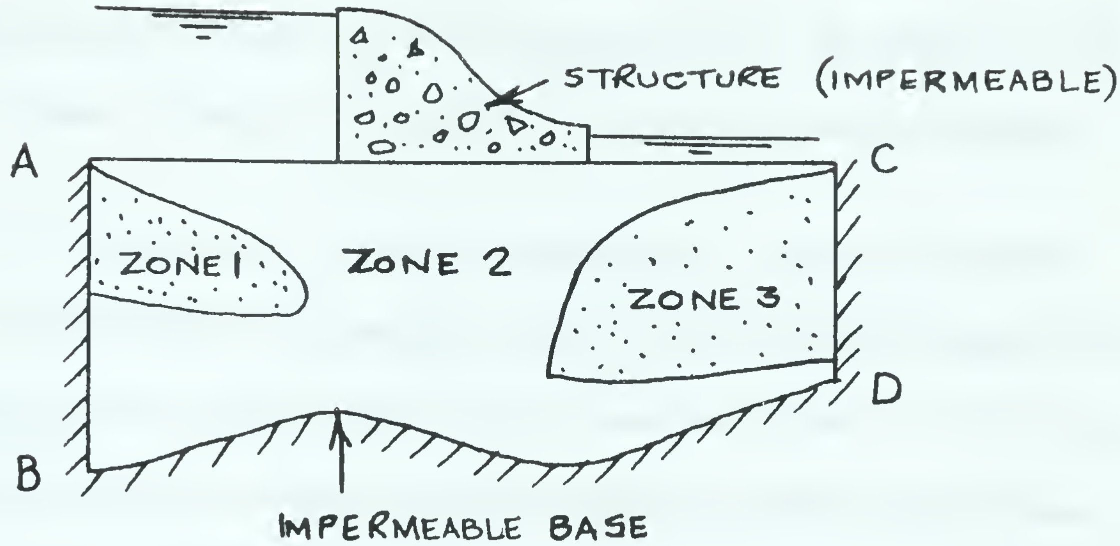
(b)

FIGURE 2.2. Examples of the square relaxation mesh.

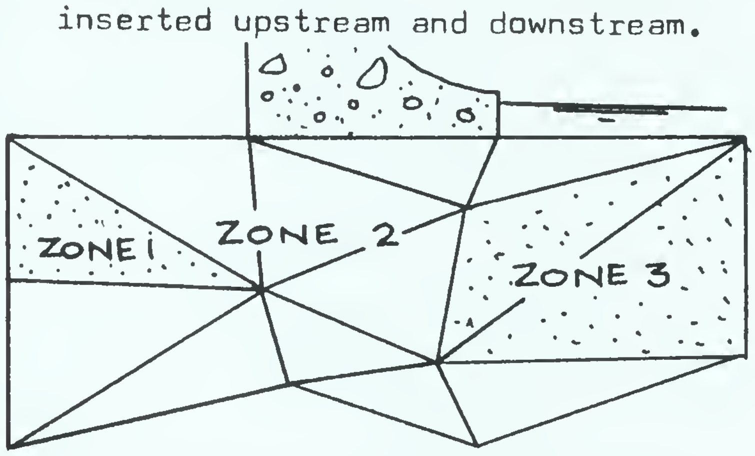
mentioned (SECTION 2.2). Inspection of equations (2.2a) and (2.2b) indicates that nodes not adjacent to boundaries are no cause for difficulty, but nodes adjacent to boundaries require individual treatment; thus if the boundaries are not approximated along the mesh sides, the method becomes appreciably more laborious if being performed by hand, or, if an electronic computer is being used, so much more programming is required.

Gibson and Lumb (1953) appear to have reduced the difficulty of nodes not lying on the boundaries; in an analysis of an earth dam, the shape was transformed so that the base angles became 45°; a square mesh, with nodes lying on the inclined faces, was used. Scott (1963, chapter 4) has suggested that, for the case of a triangular dam, a mesh consisting of equilateral triangles might advantageously be used. By transforming the vertical scale of the dam, the faces could be arranged to lie at 60° to the horizontal. The ensuing computations would then be less laborious.

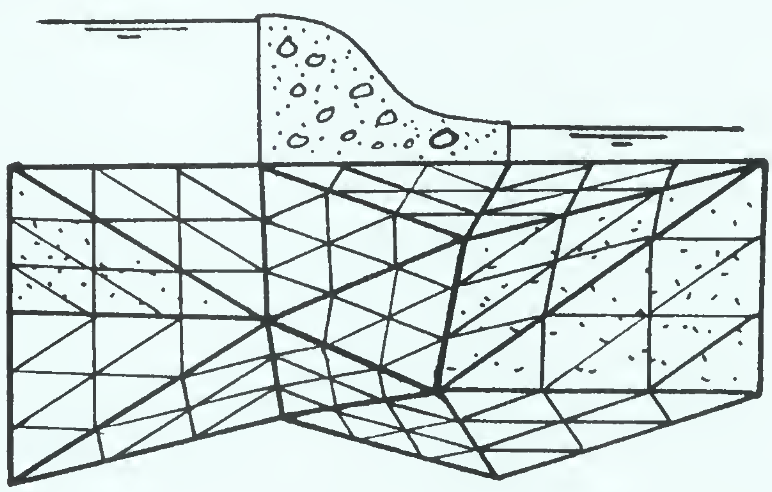
The writer's proposal for dealing with the problem of flow through a non-homogeneous, anisotropic soil was to take Scott's suggestion of a triangular mesh even further. Having approximated any curved boundaries that are present in the flow region to a series of straight lines, the section is divided into triangles so that each zone boundary lies along a side of one of the triangles. The flow region can then be regarded as looking like a jig-saw puzzle in which the pieces consist of a variety of dissimilar triangles. Each triangle is then sub-divided into smaller, similar triangles by drawing equally spaced lines parallel to each of the sides. The procedure is illustrated in FIGURE 2.3. By using this procedure for forming a mesh, the subsequent numerical work is systematized, since the positions of nodes always fall into one of three categories, as follows: (a) inside a large triangle and not less than



(a) Flow region with arbitary boundaries AB and CD inserted upstream and downstream.



(b) Flow region divided into dissimilar triangles.



(c) Mesh configuration; for clarity, each triangle is shown sub-divided into only nine smaller triangles; actually there may be any number of sub-divisions.

FIGURE 2.3. Procedure for forming mesh.

one mesh length distant from any of its edges, (b) on an edge, but not at an apex, of a large triangle, or (c) at an apex of a large triangle.

As is indicated in the next chapter, the finite difference equations associated with the proposed type of mesh are more complicated than the equation used with a square mesh (equation 2.2a). It is this additional complexity that makes a hand solution extremely laborious, but causes little extra difficulty when a computer is available.

CHAPTER III

DERIVATION OF FINITE DIFFERENCE EQUATIONS

3.1 Introduction

Most textbooks which give the derivation of the basic equation of flow do so by considering an elemental rectangle, or rectangular prism in the case of three-dimensional flow, having sides parallel to the directions of principal permeability. The derivation uses Darcy's law to obtain an expression for the flow across each face of the rectangle in terms of the hydraulic gradient; the algebraic sum of these expressions is made equal to zero in order to satisfy the continuity condition. As explained in SECTION 2.4, a triangular mesh is adopted herein, so that difficulties at zone boundaries are avoided; this mesh configuration has three axes, each parallel to one of the sides of the triangle. The angle between each axis and some datum direction has any general value, making it advantageous to use a circle instead of a rectangle for the purpose of deriving a continuity equation. If this circle is centred on the intersection of the axes, each axis becomes a radius and some symmetry is retained, making the algebra required in the derivation somewhat easier than if a rectangle had been used. Apart from Darcy's law and the continuity condition, no additional physical concepts are introduced in the derivation of the continuity equation in terms of three axes in a plane.

The finite difference equations are a special form of the continuity equation; they are partly obtained through approximate expressions for the

derivatives of head with respect to distance. The equations always appear herein in a form relating the head at an arbitary node 0 to the adjacent nodes on all the mesh axes through 0.

The following sections describe the principal steps in the derivation of the continuity equation and the finite difference equations. Greater detail of the derivations is presented in appendix A (continuity equation) and appendix B (finite difference equations).

3.2 Continuity equation

The finite difference equations for each category of node cannot be obtained until the continuity equation in terms of three axes in a plane has been derived.

Consider the elemental circle centred on an arbitary point 0 in an anisotropic soil having principal permeabilities $k_{x,y}$ in the x- and y- directions respectively (FIGURE 3.1). By consideration of rates of change of superficial velocity (hereinafter called simply 'velocity'), an expression is obtained for the outward flow q across the elemental arc PP', in terms of the velocity components $v_{x,y}$ at 0 in the x- and y- directions respectively, as follows[1]:

$$\left.\begin{aligned} q = &\{ v_x \cos\alpha + v_y \sin\alpha \} . \Delta s . \delta\alpha \\ &+ \{ (v_x)_s \cos\alpha + (v_y)_s \sin\alpha \} . (\Delta s)^2 . \delta\alpha \end{aligned}\right\} \quad (3.1)$$

Equation (3.1) can be altered to equation (3.3)[2] by use of Darcy's law.

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[1]See footnote, appendix A, page A2.
[2]Equations in this chapter are numbered to agree with appendices A and B, in which they are numbered consecutively.

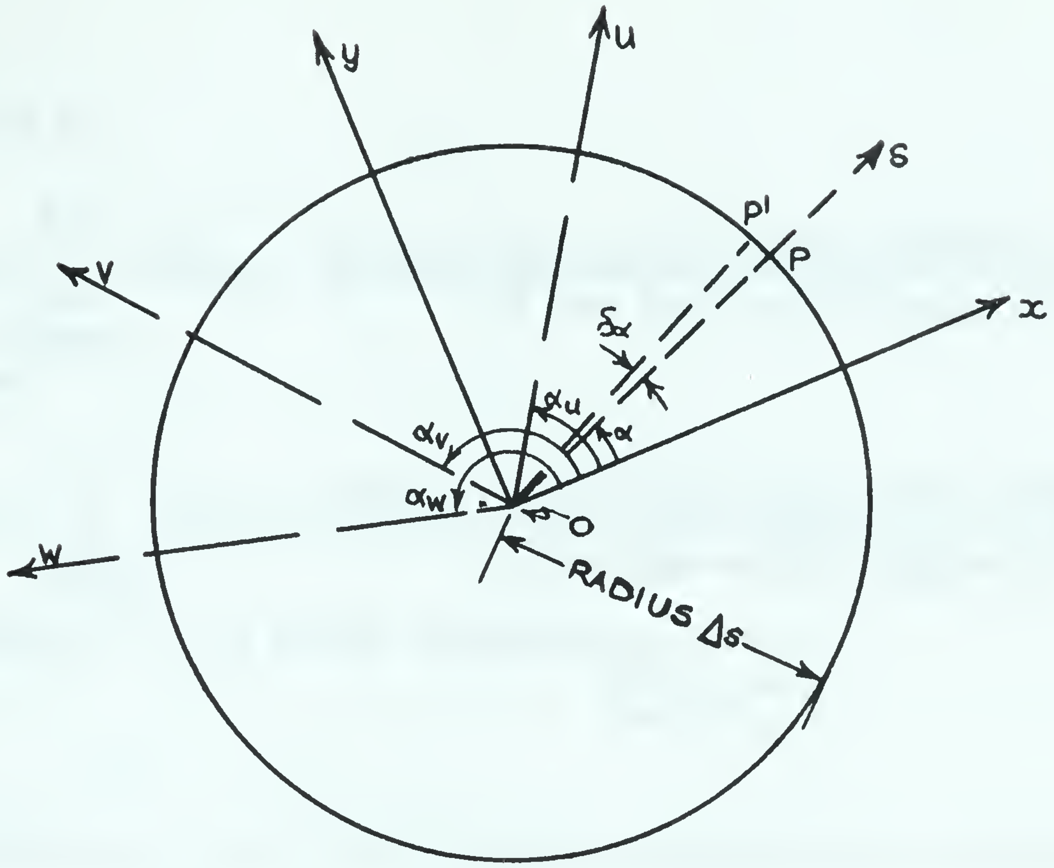


FIGURE 3.1. Elemental circle.

$$q = \left\{ v_x \cos\alpha + v_y \sin\alpha \right\} . \Delta s . \delta\alpha - \left\{ k_x (h)_{xs} \cos\alpha + k_y (h)_{ys} \sin\alpha \right\} . (\Delta s)^2 . \delta\alpha \tag{3.3}$$

The continuity equation is conveniently expressed in terms of the three axes Ou, Ov and Ow. The terms $(h)_{xs}$ and $(h)_{ys}$ in equation (3.3) can be expressed in terms of the second derivatives of h along any two of the Ou, Ov and Ow axes. The third axis is used to eliminate the term comprising the mixed second derivative of h along the first two axes. For example, if Ov is chosen as the third axis, $(h)_{xs}$ and $(h)_{ys}$ are expressed as functions of $(h)_{uu}$, $(h)_{ww}$ and $(h)_{uw}$. Another expression relating $(h)_{vv}$, $(h)_{uu}$, $(h)_{ww}$ and $(h)_{uw}$ is used to eliminate $(h)_{uw}$, thus leaving $(h)_{xs}$ and $(h)_{ys}$ expressed in terms of $(h)_{uu}$, $(h)_{vv}$ and $(h)_{ww}$. As would be expected, these expressions are symmetrical in u, v and w and may be written in the convenient forms of equations

(3.9c) and (3.9d).

$$(h)_{xs} = \sum_{M=u,v,w} (h)_{MM} \cdot \frac{\sin\alpha_N . \sin(\alpha-\alpha_P) + \sin\alpha_P . \sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M).\sin(\alpha_M-\alpha_P)}, \quad (3.9c)$$

and

$$(h)_{ys} = -\sum_{M=u,v,w} (h)_{MM} \cdot \frac{\cos\alpha_N . \sin(\alpha-\alpha_P) + \cos\alpha_P \sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M).\sin(\alpha_M-\alpha_P)}, \quad (3.9d)$$

where M, N and P are, respectively, u, v, w,
and v, w, u,
and w, u, v.

Equations (3.9c) and (3.9d) may be substituted in equation (3.3), which is then rearranged to give equation (3.10).

$$\left.\begin{aligned} q = & \left\{V_x\cos\alpha + V_y\sin\alpha\right\}.\Delta s.\delta\alpha \\ & -\left\{\sum(h)_{MM}\left[\frac{(k_x+k_y)\sin(\alpha_N+\alpha_P)\cos\alpha\sin\alpha - 2k_x\sin\alpha_N\sin\alpha_P\cos^2\alpha - 2k_y\cos\alpha_N\cos\alpha_P\sin^2\alpha}{2\sin(\alpha_N-\alpha_M)\sin(\alpha_M-\alpha_P)}\right]\right\}(\Delta s)^2.\delta\alpha \end{aligned}\right\} \quad (3.10)$$

Continuity requires that the net outflow from the circle is zero when O is a point inside a homogeneous soil. This condition leads to equation (3.11).

$$\sum_{M=u,v,w} (h)_{MM} \cdot \frac{k_x \sin\alpha_N \sin\alpha_P + k_y \cos\alpha_N \cos\alpha_P}{\sin(\alpha_M-\alpha_N).\sin(\alpha_M-\alpha_P)} = 0. \quad (3.11)$$

Equation (3.11) is the continuity equation of flow, in terms of three axes in a plane.

3.3 Finite difference equation: point inside a homogeneous soil

For steady state flow conditions, the continuity equation is satisfied at all points in the flow region. When a mesh is superimposed on the flow region, an approximation to the steady-state flow conditions is obtained if the

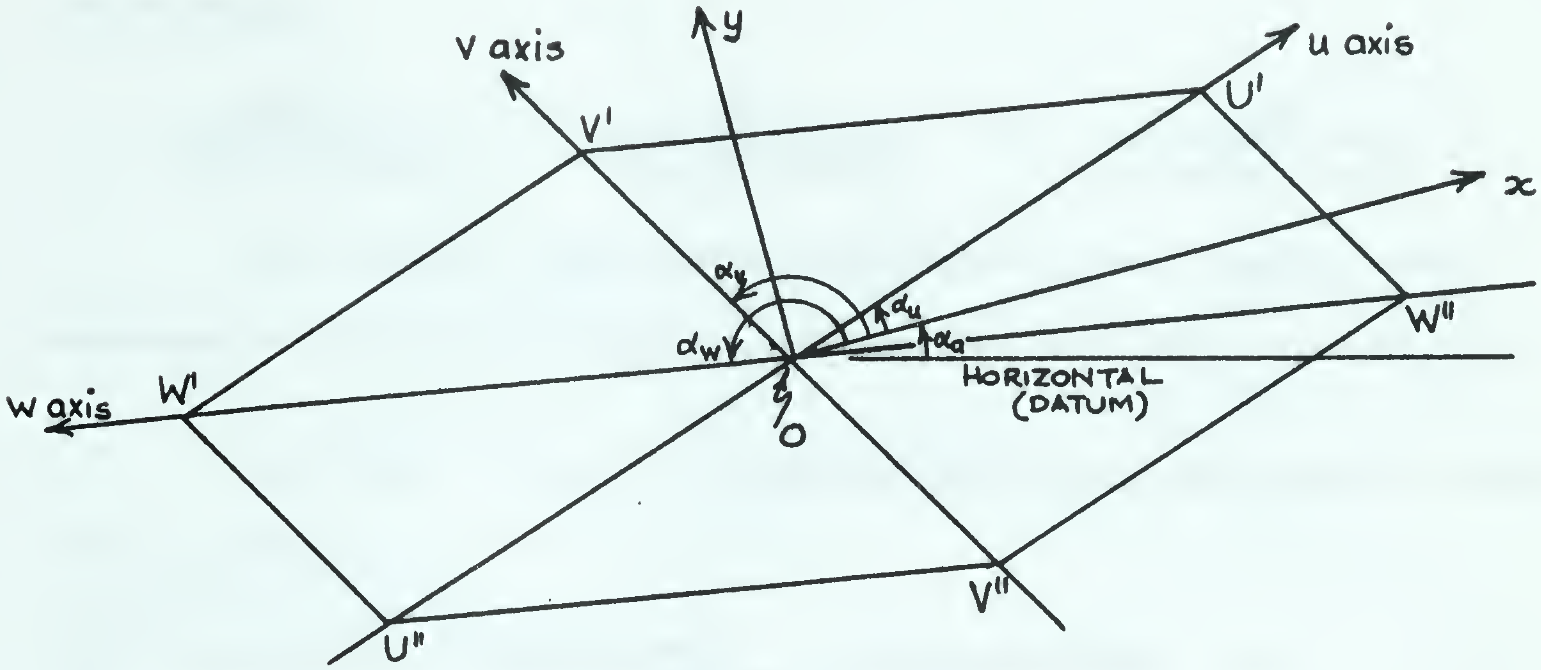


FIGURE 3.2 Part of a triangular mesh.

continuity equation is satisfied at all nodes. The difference between the approximation and the true conditions is small if a fine mesh is chosen.

The concept of a mesh is useful as a means for expressing the second derivatives of h appearing in the continuity equation (equation 3.11) in a form which is easily soluble. For example, at point O in FIGURE 3.2, the second derivative of h with respect to distance measured along the Ou axis can be shown to be approximately equal to the right-hand side of equation (3.14a).

$$(h)_{uu} = \frac{h_u' + h_u'' - 2h_o}{(\Delta_1 u)^2}, \tag{3.14a}$$

where h_u', h_u'', h_o are the total heads at U', U", O respectively, and $\Delta_1 u$ is the length OU'.

The same form of finite difference expression can be obtained for $(h)_{vv}$ and $(h)_{ww}$ at the point O.

Referring to FIGURE 3.2, it may be observed that, by the sine rule

for triangles,

$$\frac{\Delta_1 u}{\sin(\alpha_W - \alpha_V)} = \frac{\Delta_1 v}{\sin(\alpha_W - \alpha_u)} = \frac{\Delta_1 w}{\sin(\alpha_V - \alpha_u)} ;$$

thus sines of angles can be substituted for the lengths in the denominators of the finite difference expressions such as equation (3.14a).

The finite difference expressions may then be substituted in equation (3.11), giving

$$\sum_{M=u,v,w} (h'_M + h''_M - 2h_o) \cdot \frac{k_x \sin\alpha_N \sin\alpha_P + k_y \cos\alpha_N \cos\alpha_P}{\sin(\alpha_N - \alpha_P)} = 0 , \qquad (3.17)$$

which can be written as

$$C_u . h'_u + C_v . h'_v + C_w . h'_w + C_u . h''_u + C_v . h''_v + C_w . h''_w - 2(C_u + C_v + C_w) h_o = 0, \qquad (3.18)$$

where

$$C_u = \frac{k_x \sin(\alpha'_v - \alpha_a)\sin(\alpha'_w - \alpha_a) + k_y \cos(\alpha'_v - \alpha_a)\cos(\alpha'_w - \alpha_a)}{\sin(\alpha'_v - \alpha'_w)}, \qquad (3.19a)$$

$$C_v = \frac{k_x \sin(\alpha'_w - \alpha_a)\sin(\alpha'_u - \alpha_a) + k_y \cos(\alpha'_w - \alpha_a)\cos(\alpha'_u - \alpha_a)}{\sin(\alpha'_w - \alpha'_u)}, \qquad (3.19b)$$

$$C_w = \frac{k_x \sin(\alpha'_u - \alpha_a)\sin(\alpha'_v - \alpha_a) + k_y \cos(\alpha'_u - \alpha_a)\cos(\alpha'_v - \alpha_a)}{\sin(\alpha'_w - \alpha'_u)}, \qquad (3.19c)$$

and α'_u, α'_v, α'_w, are the angles measured in a counter-clockwise sense from the datum (horizontal) direction to the axes Ou, Ov, Ow respectively.

The coefficients C_u, C_v, and C_w may be computed for any given data, and h_o may be written in terms of the heads at surrounding nodes by using equation (3.18).

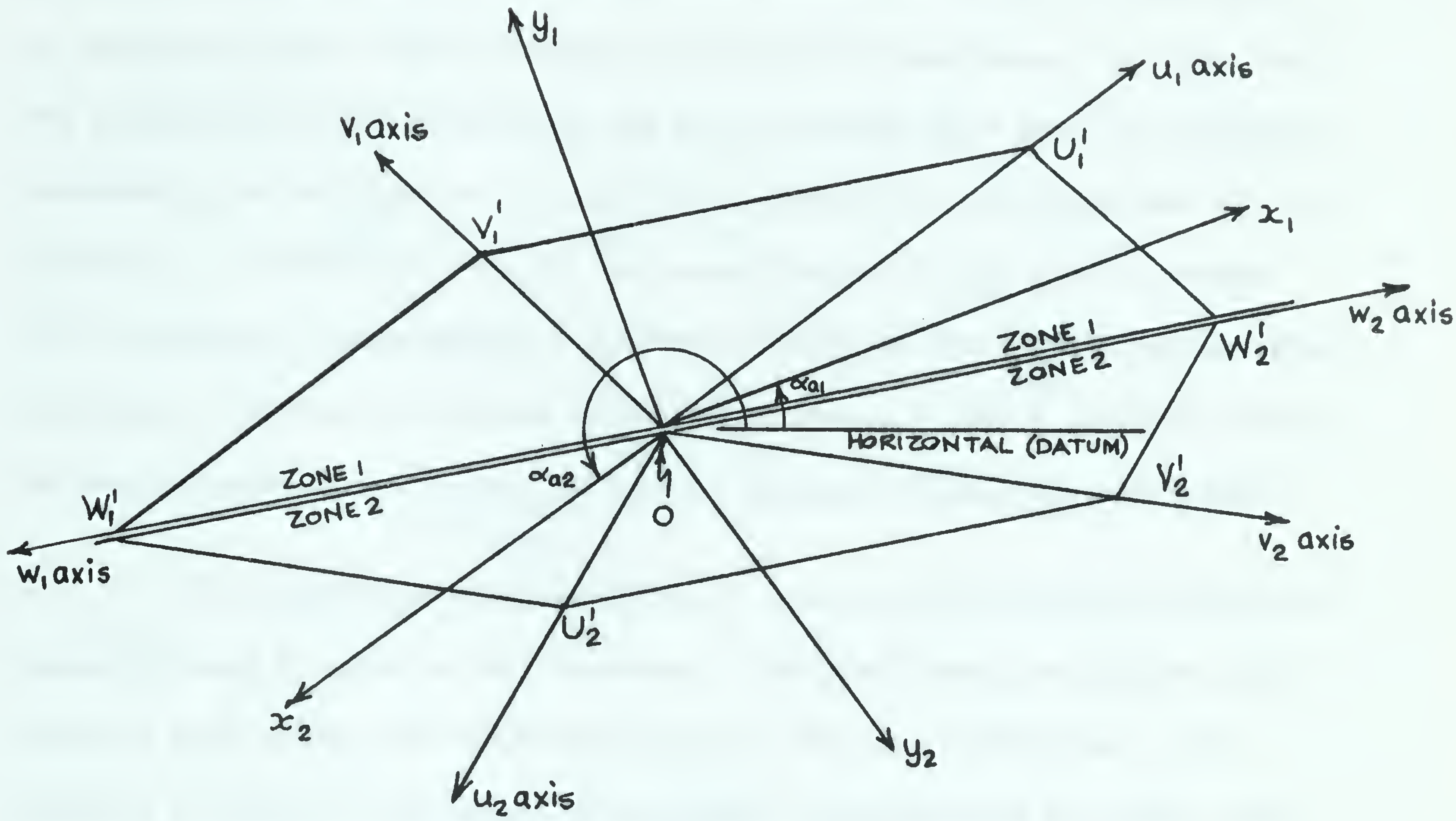


FIGURE 3.3. Point on a boundary.

3.4 Finite difference equation: point on a boundary between two zones

All nodes on common boundaries of two zones are common to the meshes of both zones, because the sides of each triangle formed by dividing the polygonal zones are sub-divided equally into the same number of segments. A boundary is any one of the Ou, Ov or Ow axes in terms of the mesh configuration of one of the zones; it is also any one of the Ou, Ov or Ow axes in terms of the other zone. FIGURE 3.3 shows the Ow axes of each zone lying along the boundary (depicted by the double line). The derivation appearing in this section applies to the case as shown.

The axes which do not lie on the boundary (i.e., the Ou and Ov axes in FIGURE 3.3) do not generally have the same alignment either side of the

boundary, and therefore a finite difference expression having the same form as equation (3.14a) cannot be stated directly for these axes. In order that the concept of finite differences may still be used, each zone is considered separately, and an image of the mesh is produced on the opposite side of the boundary. In FIGURE 3.4, one of the zones shown in FIGURE 3.3 is redrawn, with its image. The subscript 1 denotes the zone to which the notation refers. Ficticious heads are introduced at the image points U_1^f and V_1^f so that finite difference expressions similar in form to equation (3.14a) can be stated.

The ficticious heads at U_1^f and V_1^f and the corresponding ficticious heads for zone 2 comprise four unknowns. Five simultaneous equations, each defining some independent physical aspect of the flow conditions in the vicinity of point O, are therefore required; four equations are used in the elimination of the four unknowns, and the remaining one becomes the finite difference equation. The simultaneous equations are obtained from the following conditions:

(a) the continuity of flow within the elemental semi-circle GH centred on O, in zone 1, (FIGURE 3.4),

(b) the continuity of flow within a similar semi-circle in zone 2,

(c) the fact that only two axes are required to locate a point in a plane and therefore a distance measured along any one of the Ou_1, Ov_1 or Ow_1 axes may be regarded as a function of the other two axes,

(d) the same as (c), but using the Ou_2, Ov_2 and Ow_2 axes,

(e) the flow into zone 1 across GH is equal to the flow out of zone 2 across GH.

The continuity of flow within the semi-circle GH centred on O is

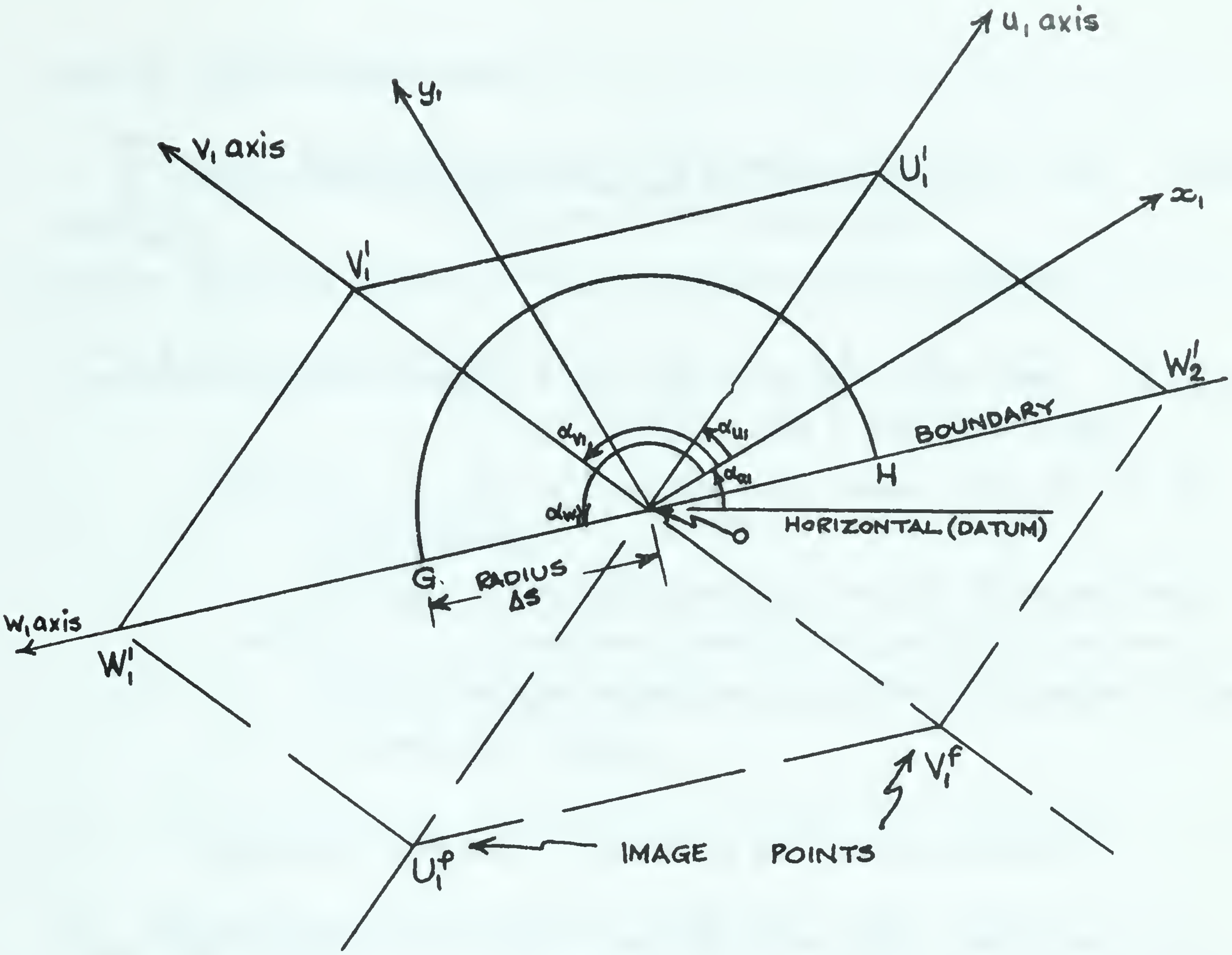


FIGURE 3.4 Image of zone 1.

satisfied if the outward flow Q_1 across the curved portion of the perimeter is equal to the flow Q_2 into zone 1 across GH. Integration of equation (3.10) between limits of $(\alpha_{w1} - \pi)$ and α_{w1} shows that

$$\left.\begin{aligned} Q_1 = {} & 2\left\{v_{x1} \sin\alpha_{w1} - v_{y1}\cos\alpha_{w1}\right\}.\Delta s \\ & + \frac{\pi(\Delta s)^2}{2} . \sum_{M_1 = u_1, v_1, w_1} (h)_{M M_1} . \frac{k_{x1}\sin\alpha_{N1}\sin\alpha_{P1} + k_{y1}\cos\alpha_{N1}\cos\alpha_{P1}}{\sin(\alpha_{N1}-\alpha_{M1}).\sin(\alpha_{M1}-\alpha_{P1})}, \end{aligned}\right\} \quad (3.20)$$

where M, N and P have the same meaning as for equation (3.10).

The flow Q_2 across GH into zone 1 may be shown to be

$$Q_2 = 2\left\{v_{x1}\sin\alpha_{w1} - v_{y1}\cos\alpha_{w1}\right\}.\Delta s. \quad (3.22)$$

Since $Q_1 = Q_2$, it follows that

$$\sum_{M_1=u_1,v_1,w_1} (h)_{MM_1} \cdot \frac{k_{x1} \sin\alpha_{N1} \sin\alpha_{P1} + k_{y1} \cos\alpha_{N1} \cos\alpha_{P1}}{\sin(\alpha_{N1}-\alpha_{M1}) \ \sin(\alpha_{M1}-\alpha_{P1})} = 0. \tag{3.23}$$

Equation (3.23) is written in finite difference form as follows:

$$\left.\begin{aligned} C_{u1}.h'_{u1} + C_{v1}.h'_{v1} + C_{w1}.h'_{w1} + C_{u1}.h^f_{u1} + C_{v1}.h^f_{v1} + C_{w1}.\ h'_{w2} \\ -2\,(C_{u1} + C_{v1} + C_{w1})\,h_o = 0, \end{aligned}\right\} \tag{3.24}$$

where h'_{u1}, h'_{v1}, h'_{w1}, h'_{w2} are the total heads at U'_1, V'_1, W'_1, W'_2 respectively,

h^f_{u1}, h^f_{v1} are the ficticious heads at U^f_1, V^f_1 respectively,

and C_{u1}, C_{v1}, C_{w1} are the coefficients appertaining to zone 1 having the same expansions as written in equations (3.19a), (3.19b) and (3.19c).

Similarly, for zone 2, the finite difference equation is

$$\left.\begin{aligned} C_{u2}.h'_{u2} + C_{v2}.h'_{v2} + C_{w2}.h'_{w2} + C_{u2}.h^f_{u2} + C_{v2}.h^f_{v2} + C_{w2}.h'_{w1} \\ -2\,(C_{u2} + C_{v2} + C_{w2})\,h_o = 0. \end{aligned}\right\} \tag{3.25}$$

If the Ou_1 and Ow_1 axes are chosen for defining the plane of flow in zone 1, $(h)_{v1}$ can be expressed in terms of $(h)_{u1}$ and $(h)_{w1}$ as follows:

$$(h)_{v1} = (h)_{u1}.(u_1)_{v1} + (h)_{w1}.(w_1)_{v1}. \tag{3.26}$$

The derivatives of h may be written in finite difference form; for example, for $(h)_{u1}$,

$$(h)_{u1} = \frac{h'_{u1} - h^f_{u1}}{2(\Delta_1 u_1)} . \tag{3.28a}$$

Expressions such as the right-hand side of equation (3.28a) may be substituted in equation (3.26), leading to the following relationship:

$$h'_{u1} - h'_{v1} + h'_{w1} - h^f_{u1} + h^f_{v1} - h'_{w2} = 0 . \tag{3.30}$$

Equation (3.31) can similarly be obtained for zone 2,

$$h'_{u2} - h'_{v2} + h'_{w2} - h^f_{u2} + h^f_{v2} - h'_{w1} = 0. \qquad (3.31)$$

An expression for the flow into zone 1 across GH has already been obtained, (equation 3.22). By Darcy's law,

$$v_{x1} = -k_{x1}.(h)_{x1} \quad \text{and} \quad v_{y1} = -k_{y1}.(h)_{y1} \;;$$

these expressions may be substituted in equation (3.22), giving

$$Q_2 = -2\left\{k_{x1}.(h)_{x1}\sin\alpha_{w1} - k_{y1}(h)_{y1}\cos\alpha_{w1}\right\}.\Delta s. \qquad (3.36)$$

The terms $(h)_{x1}$ and $(h)_{y1}$ may be replaced by expressions containing $(h)_{u1}$ and $(h)_{w1}$, leading to equation (3.38).

$$Q_2 = \left\{(h'_{u1} - h^f_{u1}).(C_{u1} + C_{v1}) + (h'_{w1} - h'_{w2}).C_{v1}\right\}.\left\{\Delta s/\Delta_1 w_1\right\}, \qquad (3.38)$$

where C_{u1} and C_{v1} have the same meaning as for equation (3.24). A further expression for Q_2 can be developed by considering zone 2, i.e.,

$$Q_2 = -\left\{(h'_{u2} - h^f_{u2}).(C_{u2} + C_{v2}) + (h'_{w2} - h'_{w1}).C_{v2}\right\}.\left\{\Delta s/\Delta_1 w_2\right\}, \qquad (3.39)$$

where C_{u2} and C_{v2} have the same meaning as for equation (3.25). Since $\Delta_1 w_1 = \Delta_1 w_2$, it follows that

$$\left.\begin{aligned}(C_{u1} + C_{v1}).(h'_{u1} - h^f_{u1}) + (C_{u2} + C_{v2}).(h'_{u2} - h^f_{u2})&\\ + (C_{v1} - C_{v2}).(h'_{w1} - h'_{w2}) = 0.&\end{aligned}\right\} \qquad (3.40)$$

The terms h^f_{u1}, h^f_{v1}, h^f_{u2} and h^f_{v2} are eliminated from equations (3.24), (3.25), (3.30), (3.31) and (3.40), leaving equation (3.41) as the equation for h_o in terms of the heads at surrounding nodes.

$$\left.\begin{aligned}C_{u1}.h'_{u1} + C_{v1}.h'_{v1} + C_{u2}.h'_{u2} + C_{v2}.h'_{v2} + \frac{C_{w1} + C_{w2}}{2}.(h'_{w1} + h'_{w2})&\\ -(C_{u1} + C_{v1} + C_{w1} + C_{u2} + C_{v2} + C_{w2}) = 0,&\end{aligned}\right\} \qquad (3.41)$$

where the Ow axes of both zones lie along the boundary; in general, when other axes lie along the boundary the subscripts u, v and w require interchanging appropriately.

Equation (3.41) may be interpreted as follows: multiply the heads at adjacent nodes on the boundary by half of the sum of the coefficients (calculated from equations 3.19a, b, c) relevant to the axes on the boundary, and add this amount to the sum of the heads at the four other adjacent nodes, each multiplied by the appropriate coefficient; the result is equal to the sum of all the coefficients multiplied by the head at point O.

The finite difference equation for a node on an impermeable boundary of the flow region is easily obtained from equation (3.41). An impermeable boundary may be regarded as a boundary between two zones, one of which has zero permeability. If the zones are denoted by 1 and 2, and if zone 2 is impermeable, inspection of equations (3.19a, b, c) shows that

$$C_{u2} = C_{v2} = C_{w2} = 0 ,$$

in equation (3.41). Thus omitting the subscript 1, it follows from equation (3.41) that, for a node on an impermeable boundary,

$$C_u.h'_u + C_v.h'_v + C_w.\frac{h'_w + h''_w}{2} - (C_u + C_v + C_w).h_o = 0, \quad (3.42)$$

where h'_w, h''_w represent the heads at nodes on the boundary either side of the node at O.

3.5 Finite difference equation: point at an apex of several triangles

In setting up the mesh for a non-homogeneous soil, each polygonal zone is divided into triangles by joining certain pairs of its apices. Since the polygons are irregular in shape and the number of sides of each polygon

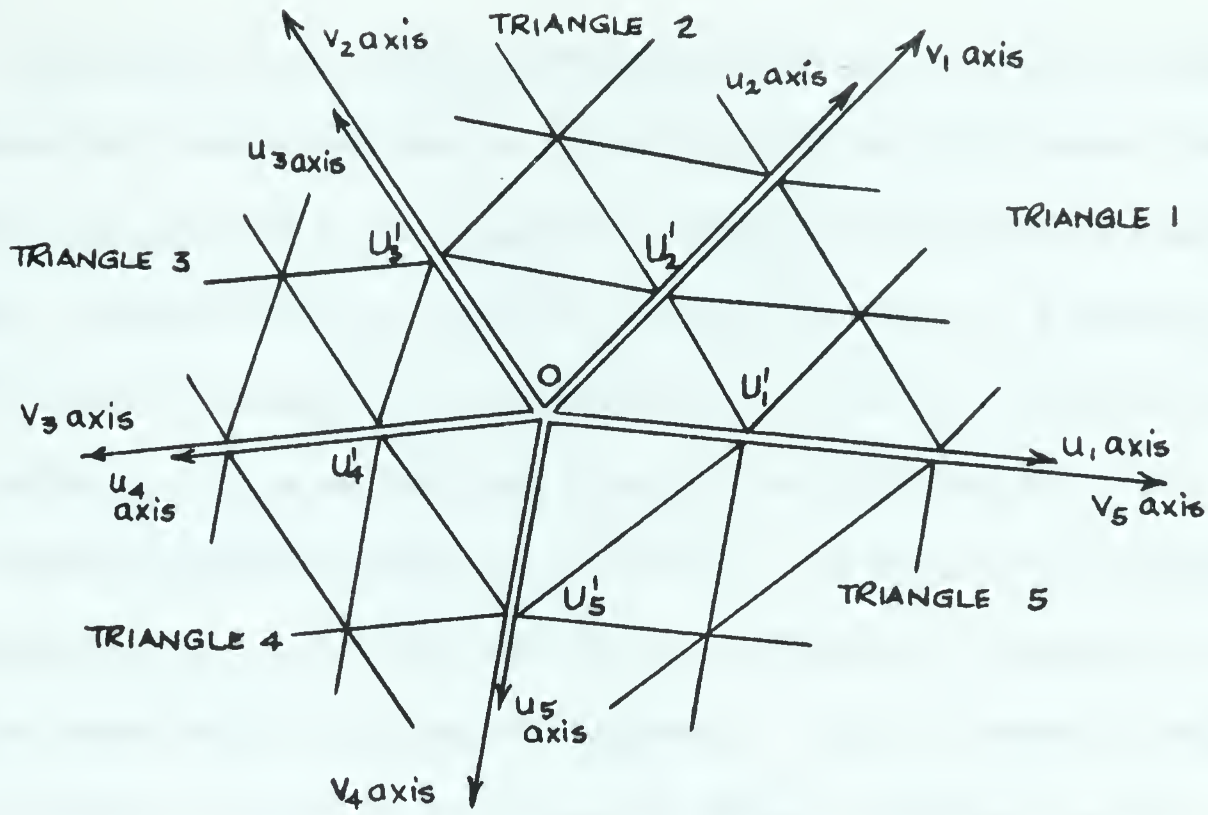


FIGURE 3.5. The apex of several triangles.

is not a constant, each apex of any resulting triangle may be common to several triangles, as shown in FIGURE 3.5.

Despite considerable effort, a rigorous proof of the finite difference equation for the head at an apex of several zones was not found. The most promising approach used by the writer to establish such an equation was similar to the method used for obtaining the finite difference equation for a point on the boundary between two zones. An elemental circle was drawn with its centre at O and equations for the continuity condition for the sector of the circle lying in each triangle were determined in a finite difference form, using ficticious heads at image points. Also determined were the flow conditions across each boundary, and equations based on the fact that a measurement along one axis of any triangle is a function of the other two axes of that triangle. These conditions were insufficient to yield an equation of the desired form, and the writer's attempts to establish such an equation by rigorous methods were abandoned in favour of a more heuristic approach.

Inspection of the finite difference equations for points inside a homogeneous soil and on boundaries (equations 3.18 and 3.41) shows that, for these cases, the value of h_0 is a weighted average of the heads at every adjacent node. Except when O is joined to the adjacent node by a boundary, the head at the node is multiplied by the coefficient C having the subscript of the same letter u, v or w as the axes through O on which the node lies. If O and the adjacent node are joined by a boundary, the head at the adjacent node is multiplied by half of the sum of the coefficients C relevant to the axes of both zones which lie along the boundary. Thus, it appears reasonable to infer that the head at O, an apex of several triangles, is the weighted average of the heads at adjacent nodes. Fox example, in FIGURE 3.5,

$$\frac{C_{u1}+C_{v5}}{2}h'_{u1}+\frac{C_{u2}+C_{v1}}{2}h'_{u2}+\frac{C_{u3}+C_{v2}}{2}h'_{u3}+\frac{C_{u4}+C_{v3}}{2}h'_{u4}+\frac{C_{u5}+C_{v4}}{2}h'_{u5}$$
$$-\left\{\frac{C_{u1}+C_{u2}+C_{u3}+C_{u4}+C_{u5}+C_{v1}+C_{v2}+C_{v3}+C_{v4}+C_{v5}}{2}\right\}h_o=0,$$

where the subscripts 1, 2, 3, 4, 5 denote the triangle to which the notation refers.

More generally, if there are n triangles with apices at O,[1]

$$\left.\begin{aligned}(C_{u1}+C_{vn})h'_{u1}+(C_{u2}+C_{v1})h'_{u2}+\cdots\cdots+(C_{un}+C_{vn-1})h'_{un}\\-(C_{u1}+C_{v1}+C_{u2}+C_{v2}+\cdots+C_{un}+C_{vn})h_o=0.\end{aligned}\right\}\quad(3.43)$$

If the common apex of several triangles lies on an impermeable boundary, as shown in FIGURE 3.6, the finite difference equation is obtained from equation (3.43) by a similar approach to that used for obtaining equation (3.42). An additional zone of zero permeability is introduced in the triangular area between the impermeable boundaries. If these boundaries are located next to triangles 1 and n, equation (3.43) reduces to:

[1] After this thesis was completed, a derivation of the finite difference equation (3.43) was obtained by the writer, (see Appendix F).

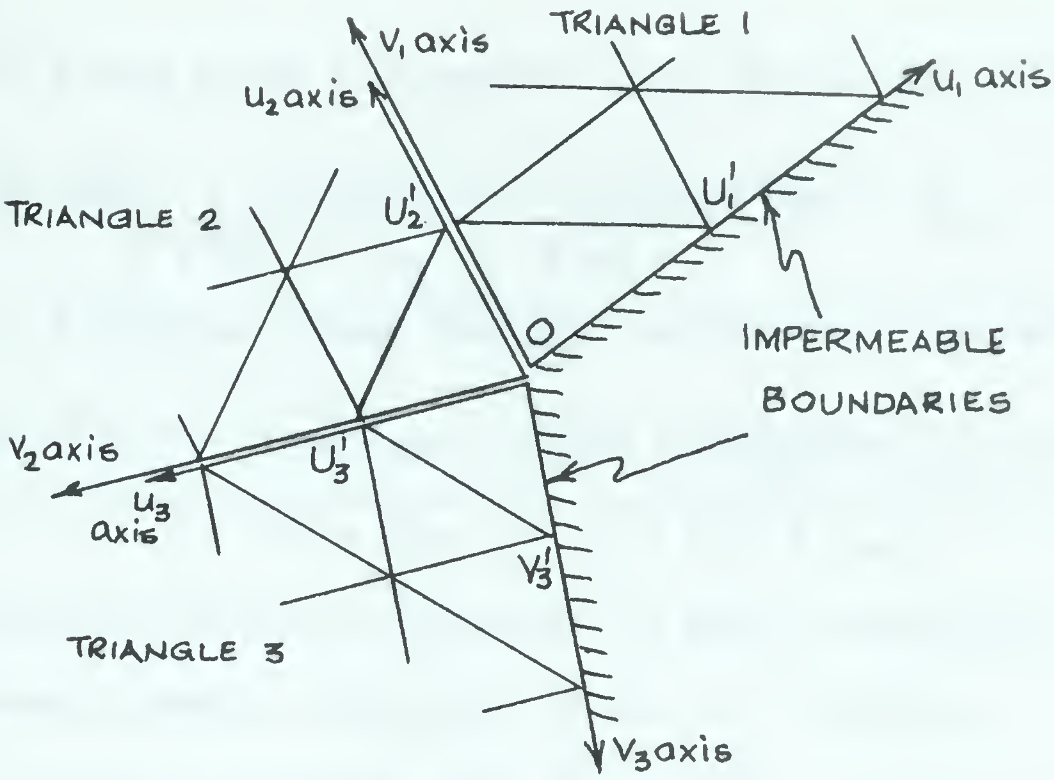


FIGURE 3.6. An apex of several triangles at an impermeable boundary

$$\left.\begin{aligned} C_{u1}.h'_{u1} + (C_{u2}+C_{v1})h'_{u2} + (C_{u3}+C_{v2})h'_{u3} + \ldots + (C_{un}+C_{vn-1})h'_{un} + C_{vn}.h'_{vn} \\ - (C_{u1}+C_{v1}+C_{u2}+C_{v2}+\ldots+C_{un}+C_{vn})h_o = 0. \end{aligned}\right\} \quad (3.44)$$

Equations (3.43) and (3.44) apply to cases where the Ou and Ov axes of each triangle pass through O. If some of the triangles have their Ov and Ow axes passing through O, the subscripts u and v for those triangles are changed to v and w respectively. Similarly, for any triangles having their Ow and Ou axes passing through O, the subscripts u and v are changed to w and u.

3.6 Equations for the residuals

The finite difference equations derived in SECTIONS 3.3, 3.4 and 3.5 must be satisfied at all appropriate nodes for steady-state conditions to exist. It has already been stated that initially a guess is made for the head at each node; hence the left-hand sides of the finite difference equations are not usually equal to zero. The following equations, which follow from equations (3.18), (3.41), (3.42), (3.43) and (3.44), make allowance for this difference. It appears in the form of a residual, denoted by the letter R.

For a node inside a homogeneous soil, the residual R is given by

$$R = \frac{C_u.(h_u' + h_u'') + C_v.(h_v' + h_v'') + C_w.(h_w' + h_w'')}{2\,(C_u + C_v + C_w)} - h_o . \tag{3.45}$$

For a node on a boundary between two zones or triangles numbered 1 and 2,

$$R = \frac{C_{u1}.h_{u1}' + C_{v1}.h_{v1}' + C_{u2}.h_{u2}' + C_{v2}.h_{v2}' + \frac{C_{w1}+C_{w2}}{2}.(h_{w1}' + h_{w2}')}{C_{u1} + C_{v1} + C_{w1} + C_{u2} + C_{v2} + C_{w2}}, \tag{3.46}$$

where the Ow axes of both zones lie along the boundary. For a node on an impermeable boundary, but not at an apex of a triangle,

$$R = \frac{C_u.h_u' + C_v.h_v' + C_w.\frac{h_w' + h_w''}{2}}{C_u + C_v + C_w} - h_o , \tag{3.47}$$

where the Ow axis lies along the boundary. For a node at the apex of n triangles, numbered 1 to n, not located on a boundary of the flow region,

$$R = \frac{(C_{u1}+C_{vn}).h_{u1}' + (C_{u2}+C_{v1})h_{u2}' + \dots + (C_{un} + C_{vn-1})h_{un}'}{C_{u1} + C_{v1} + C_{u2} + C_{v2} + \dots + C_{un} + C_{vn}} - h_o , \tag{3.48}$$

where the Ou and Ov axes of each triangle pass through the node. For a node at the apex of n triangles, number 1 to n, and also located on impermeable boundaries,

$$R = \frac{C_{u1}.h_{u1}' + (C_{u2}+C_{v1}).h_{u2}' + (C_{u3}+C_{v2})h_{u3}' + \dots + (C_{un}+C_{vn-1})h_{un}' + C_{vn}.h_{vn}'}{C_{u1} + C_{v1} + C_{u2} + C_{v2} + \dots + C_{un} + C_{vn}} - h_o , \tag{3.49}$$

where the Ou and Ov axes of each triangle pass through the node, and the impermeable boundaries are located next to triangles 1 and n. Along a boundary across which water passes into or out of the flow region, the total head is constant, and can be computed from the external conditions. Thus the heads at all nodes located on such a boundary are known at the start of the analysis, and, because they are not altered during the subsequent computations, the residual at these nodes is zero, i.e.

$$R = 0 . \tag{3.50}$$

The computation and manipulation of residuals at every node in the flow region, is performed by the computer.

CHAPTER IV

COMPUTER PROGRAM ASSEMBLY

4.1 The functions of the computer program

The computer program is, in effect, a list of instructions which causes the computer to perform certain operations and arrive at a meaningful result. The sequence of operations is the same for different sets of data, until a conditional instruction is reached; the execution of a conditional instruction is dependent upon some result obtained earlier in the computation. Proper use of conditional instructions, therefore, enables the same program to be used for sets of data requiring difference sequences of operations[1].

For most problems that call for the use of a computer, it is desirable that the data be presented in its basic form, and that the results appear in a readily usable form; in this way, the computer is doing as much work as possible. The basic data for a seepage problem are the minimum and maximum permeabilities for each zone and their directions, boundary locations and other such items. The program is designed so that all the remaining work, including the division of the zones into triangles, the calculation and storage of the coefficients 'C' in equations (3.45) to (3.49), and the numerical approximations, is performed by the computer.

[1] No attempt is made herein to describe the basic techniques of computer programming or the functional components of the computer. For this information, reference should be made to the IBM manual on engineering analysis for computers (1963).

4.2 Description of the computer program

A flow chart showing diagrammatically the sequence of operations performed by the computer is presented in FIGURE 4.1.

The data is arranged so that all of the information on one zone is received and the coefficients 'C' calculated for each triangle within the zone before going on to the next zone, (steps 3 to 9, FIGURE 4.1). In this way, the values of the minimum and maximum permeabilities and the direction of the minimum permeability may be dispensed with as soon as the coefficients 'C' have been calculated. The coordinates of the corner points of the zone are stored as apices of triangles, and the same program statements are then used when considering the next zone. The procedure specified for steps 6, 7 and 8, dividing the zones into triangles, is fully described in appendix C.

In the data for each zone the maximum desirable distance between nodes is specified. This distance may vary from node to node within the flow region. Since the sides of every triangle are sub-divided into the same number of segments, this number is determined by the zone which has the maximum value of the ratio:

$$\frac{\text{longest side of triangle in zone}}{\text{maximum distance between nodes for zone}}$$

Thus a triangle whose longest side is 10 ft. and in which the node spacing is a maximum of 2 ft. is the criterion rather than a 60 ft. triangle in which a node spacing of 15 ft. is satisfactory. The decision to have large node spacings in some zones and small node spacings elsewhere should be made cautiously, because allowable errors arising from the coarse mesh are likely

1. Start.

2. Input NZ, the number of zones.

From 50

3. Input the permeability data and direction of minimum permeability for one zone, and output these items in tabular form.

4. Input the coordinates of corner points of this zone in clockwise order.

5. Calculate directions of sides of the polygon forming this zone.

6. Join two alternate corner points of the polygon by a straight line, thereby creating a triangle, and reducing the number of polygon sides by one.

7. Tabulate the coordinates of the apices of this triangle in clockwise order in the arrays XU(I), YU(I), XV(I), YV(I), XW(I), YW(I) respectively and calculate the coefficients C_u, C_v, C_w to be used in equations (3.45) to (3.49).

8. Repeat steps 6 and 7 until the number of polygon sides is reduced to three.

9. Calculate the desirable number of sub-divisions of each side within this zone so that the distance between nodes is less than a pre-stated maximum for the zone.

10. Repeat steps 3 to 9 for each zone, i.e., NZ times.

11. Obtain N (desirable number of sub-divisions throughout the flow region) from step 9 for each zone. If N $>$ 11, replace it by 11.

12. Input coordinates locating a face of constant head (potential) and HA, the head acting thereon. Output these items in tabular form.

13. By sorting through the arrays XU(I),...., YW(I), find the side of the triangle which lies on the constant potential face and note the triangle number in one of the arrays IUU(J2), IVV(J3), IWW(J1). Assign the value HA to the head at all nodes situated along the face, and at nodes at apices of other triangles which contact the constant potential face at its end.

14. Repeat steps 12 and 13 until a dummy input is received.

15. Let the arithmetic mean of the values HA for each constant potential face be the initial "guess" of the heads at all nodes except those allocated in step 13.

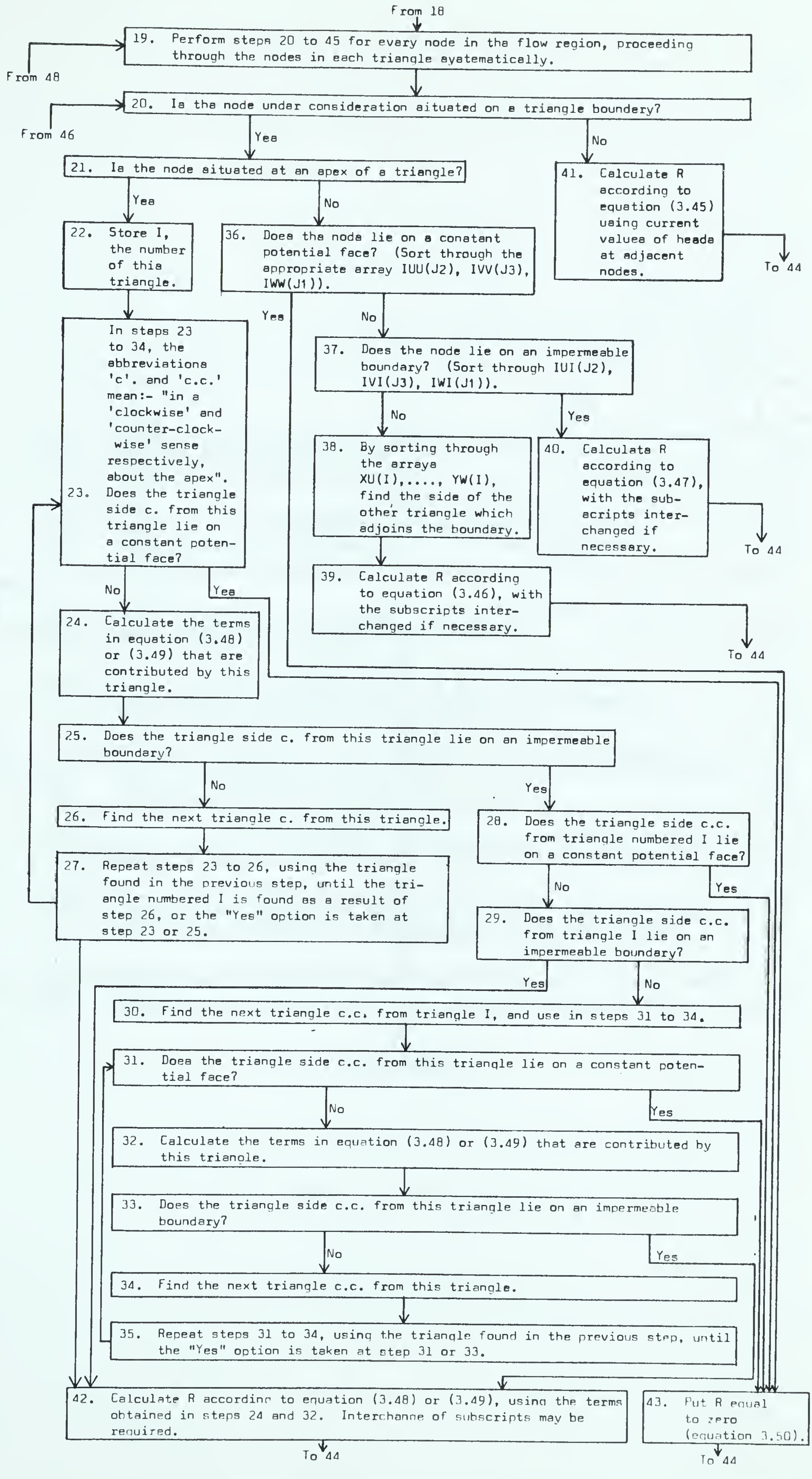
16. Input coordinates locating an impermeable boundary; output these items in tabular form.

17. Find the side of the triangle which lies on the impermeable boundary, and note the triangle number in one of the arrays IUI(J2), IVI(J3), IWI(J1).

18. Repeat steps 16 and 17 until a dummy input is received.

To 19

(FIGURE 4.1 continued on next page)



(FIGURE 4.1 continued on next page)

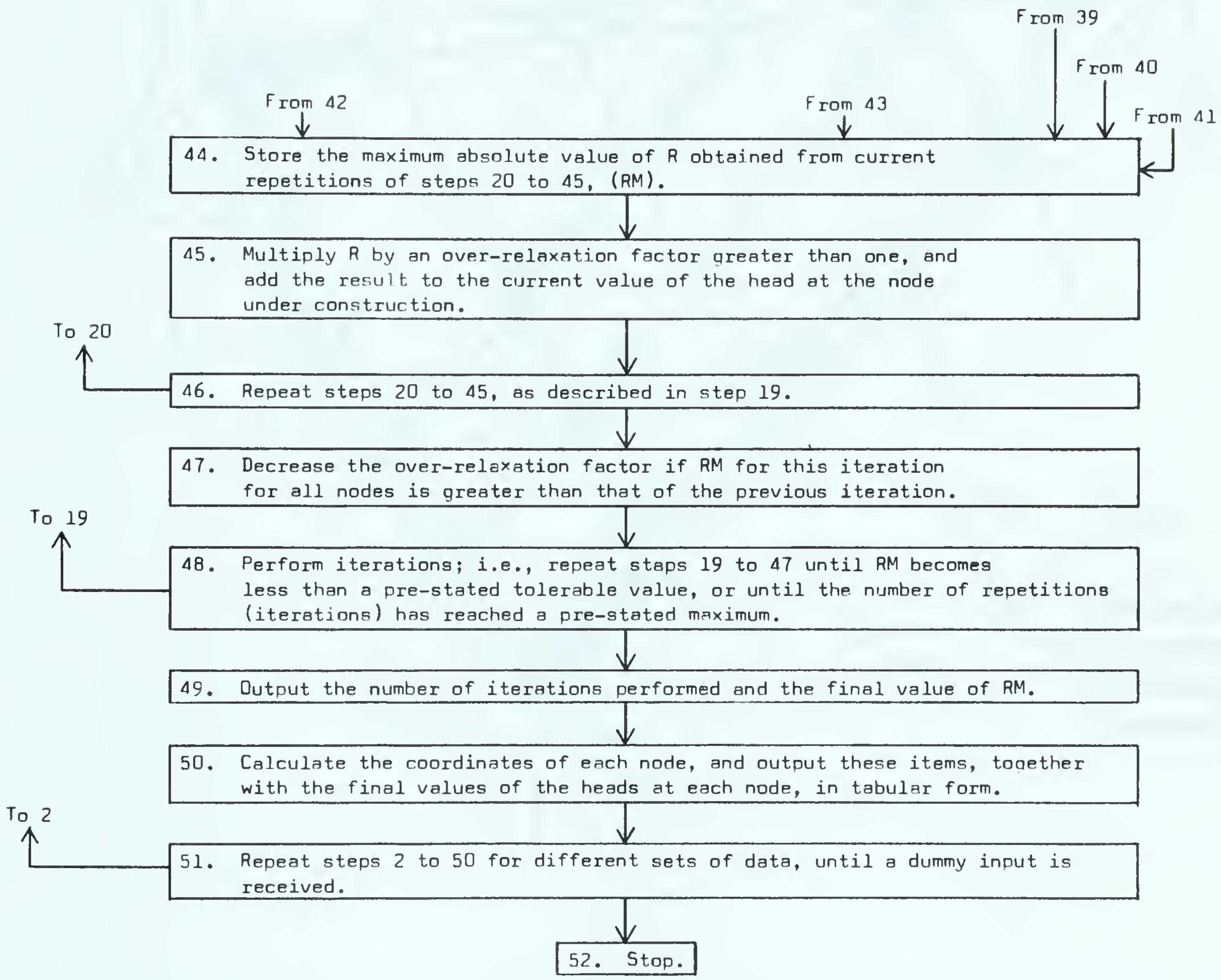


FIGURE 4.1 Flow chart

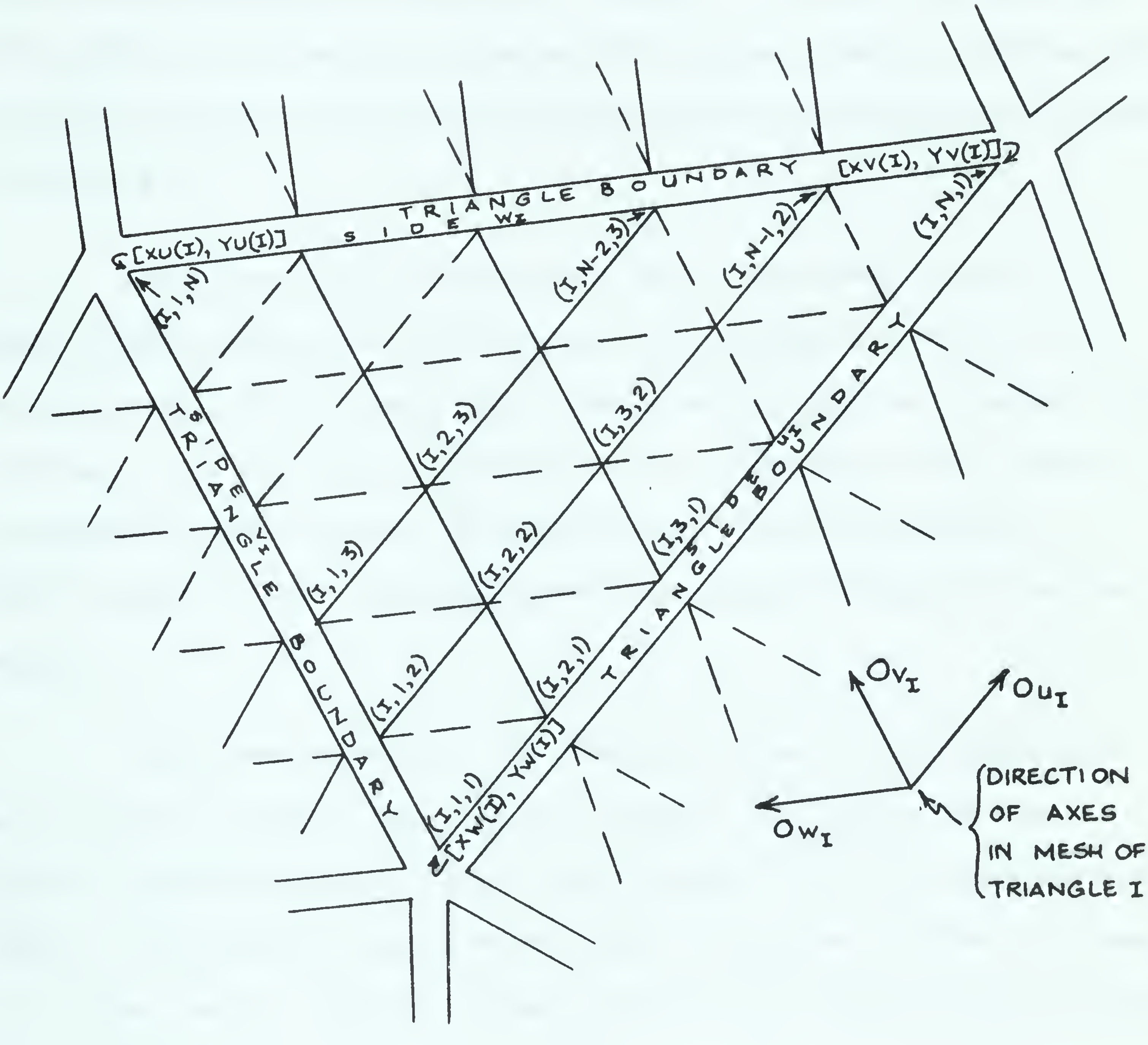


FIGURE 4.2 Numbering convention for nodes

to encroach into the fine mesh, where they may constitute serious errors; this effect is discussed in SECTION 6.1. Because the number of memory storage locations in the computer is limited, the program includes a provision that the number of nodes on the sides of any triangle shall not exceed eleven, (step 11, FIGURE 4.1).

The arrays IUU(J2), IVV(J3), and IWW(J1) are formed in step 13 in order that the determination of the value of the residual at a node on a triangle boundary is facilitated. Each triangle boundary lying on a constant potential face of the flow region is identified by the appearance of the number of its triangle in these arrays. The arrays IUI(J2), IVI(J3), and IWI(J1) in step 17 perform a similar function for the impermeable boundaries of the flow region.

The boundary conditions are completely defined and the meshes in each triangle are fixed before step 19 is reached. Any node in the flow region is conveniently identified by three integers; the first integer is the number of the triangle in which the node lies. Inspection of FIGURE 4.2 shows that a node lies on a triangle boundary if any of the following three conditions apply:

(a) the second of the three integers is 1, indicating that the Ov axis and the boundary are colinear;

(b) the third integer is 1, indicating that the Ou axis and the boundary are colinear;

(c) the sum of the second and third integers is N+1, where N is the number of nodes on a triangle side, indicating that the Ow axis and the boundary are colinear.

Steps 19 to 47 comprise one iteration, in which the residuals R are computed for every node within the flow region. For any node situated inside a triangle, R is calculated using equation (3.45). Each triangle side is common to a constant potential face, an impermeable boundary, or the side of another triangle; thus, for a node on a triangle side, the arrays IUU(J2), etc., IUI(J2), etc. and those listing the coordinates of the apices of triangles are inspected (steps 36, 37 and 38) so that the appropriate equation for calculating R can be selected. The procedure used for finding the sides of triangles which lie along a common boundary is described in appendix C.

The node at an apex of a triangle is generally common to other triangles, (FIGURE 3.5). Each of these triangles is searched for before the residual at the node can be computed. The node is identified at first by a particular triangle, which may be, for example, in FIGURE 3.5, triangle 5. The expressions ($C_{u5} \cdot h'_{u5} + C_{v5} \cdot h'_{v5}$) and ($C_{u5} + C_{v5}$) are calculated and stored for use later as components of the equation for the residual. The procedure described in an earlier paragraph for finding an adjoining triangle is used to determine the triangle which is situated next to triangle **5** progressing in a clockwise sense about the apex, (step 26). In FIGURE **3.5,** triangle 4 is revealed; the expressions ($C_{u4} \cdot h'_{u4} + C_{v4} \cdot h'_{v4}$) and ($C_{u4} + C_{v4}$) are calculated and added to the expressions already obtained. Eventually, every triangle is revealed and the residual is computed by determining the ratio of the sums of expressions obtained for each triangle, and subtracting from it the existing value of the head at the node. If, while progressing from triangle to triangle round the apex, a constant potential face is encountered, the residual is made equal to zero. If an impermeable boundary is encountered, further progression in a clockwise sense is prevented; so,

starting again at the original triangle, any remaining triangles are revealed in counter-clockwise order, (steps 28 to 35). Thus, in FIGURE 3.6, if the node is identified initially by triangle 2, the next triangle revealed is triangle 1 and then the impermeable boundary Ou_1 is encountered; proceeding next in a counter-clockwise sense from triangle 2, triangle 3 is revealed.

After the residual at a node is calculated, it is multiplied by a number greater than one, called an over-relaxation factor, and the product is added to the current value of the head to obtain a new value, (step 45). The presence of the over-relaxation factor is discussed in SECTION 6.2: its effect generally is to quicken the liquidation of the residuals, and so reduce the number of iterations. If the residuals for the nodes in an area of appreciable size are less than the pre-stated tolerable value, but are all positive or all negative, a large error may exist in the calculated heads at nodes in the centre of the area. The use of an over-relaxation factor is believed to overcome this effect, by causing the convergence of the head at any node to be an oscillating process. To prevent the oscillations becoming divergent, provision is made for reducing the over-relaxation factor as soon as any increase occurs in the absolute value of the residuals from one iteration to the next, (step 47).

Several sets of data may be analysed in the same run of a program. After the output for any set of data is completed, a statement is read which returns the control to the start of the program, and the next set of data is input and analysed. The computer stops when a dummy input is received; this input consists of a negative number or zero representing NZ, the number of zones in a flow region.

4.3 The format of the program and the data

The program is written in FORTRAN language, which has a format close to that of mathematical equations and uses certain words of the English language to describe each operation. The complete program is presented in appendix D. Each statement, or line, is punched on a standard punch card.

Built into the program are a number of statements which cause the computer to stop if the number of items of data is too large or if there is a detectable error in the data. Such an error might be that the coordinates locating constant potential faces and impermeable boundaries do not agree with the zone boundaries. When the procedure for selecting the triangle which adjoins a given triangle is being used, a set of operations is repeated using a subscript which is being incremented. Although this procedure is designed so that the triangle is always revealed before the subscript exceeds the total number of triangles in the flow region, a statement preventing the subscript increasing indefinitely is inserted into the program as a safeguard against the computer wasting time; the statement causes the computer to stop. Before any such 'unintended stop' statement is read, a message indicating the nature of the trouble is passed through output.

In order that each set of data be interpreted meaningfully, it is arranged in a definite format. The data is punched on a series of cards as indicated in appendix D. The sets of data are placed in order behind the program statements.

CHAPTER V

PRESENTATION OF WORKED EXAMPLES

5.1 Discussion of FIGURES 5.1 to 5.6

Several problems for which the computer program was used are shown in FIGURES 5.1 to 5.6. The largest residual remaining after the final iteration is shown for each case; this is equal to or greater than the tolerable maximum, depending on whether the specified number of iterations was sufficient to allow the residuals to decrease to the amount desired. The largest residual may be regarded as an indicator of the accuracy of the solution. In some cases, the solution by other methods is superimposed on the computer solution so that a direct comparison may be made.

In the example of FIGURE 5.1, the flow region consists of a homogeneous isotropic soil underlying an impermeable structure. The flow is confined by impermeable boundaries at the sides and the base, but the distance from the structure to the impermeable boundaries is large in relation to the width of the structure; thus the flow pattern at all points except near the boundaries is similar to that occurring in a semi-infinite soil, for which a mathematically exact solution is available. The complex variable theory can be used to show that the equipotential lines for the semi-infinite case are a family of hyperbolas having common foci at the corners of the structure. Inspection of FIGURE 5.1 shows that there is close agreement between the equipotential lines obtained from the computer solution and those obtained from the complex variable theory. Apart from an

expected divergence of these lines near the impermeable boundary, the greatest difference between the computer and complex variable solutions is about one percent in terms of potential head.

Before the data for the example of FIGURE 5.1 was used by the computer, the flow region was arbitarily divided into four "zones" so that the nodes were more closely spaced near the structure than elsewhere. This procedure illustrates how the node spacing may be varied within a zone by simple menipulation of the data.

FIGURE 5.2 shows the solutions from three different methods to the problem of flow round a cut-off curtain extending vertically from the centre of an impermeable structure. A separate test has shown that the equipotential lines as shown for the electric analogue solution are almost identical to those occurring in the case of an infinitely long layer of soil. Close agreement exists between the solutions obtained by the computer method and the conventional iteration method using a square mesh; the equipotential lines in each case agree to within about 3 percent in terms of the potential head values, with the greatest difference occurring near the cut-off curtain. Vertically below the cut-off curtain the solution using the square mesh is exact, because the symmetry of the flow region was used in obtaining this solution. Although symmetry could have been used in the computer solution, it was not; the error in the solution near the cut-off curtain occurs as a result of residuals not being completely liquidated. The electric analogue solution is within 10 percent of the iteration method using a square mesh, with the greatest difference occurring near the cut-off curtain. Experimental errors are believed to account for the difference between the analogue and numerical solutions.

The equipotential lines obtained by the computer method for flow through a layered soil are shown in FIGURE 5.3(a). The flow region in this example consists of two anisotropic zones and one isotropic zone, and is confined at the sides and the base by impermeable boundaries. FIGURE 5.3(b) shows the equipotential lines for the same flow region as FIGURE 5.3(a), but the data for FIGURE 5.3(b) was adjusted so that each zone was arbitarily divided into smaller "zones". There is a slight distinction between the two solutions, attributable to the different residuals and node spacings in each case.

In FIGURE 5.4, the soil is homogeneous and anisotropic with the stratification occurring at an angle of 25 degrees to the horizontal. A check was made to verify the position of the equipotential line representing 50 units of head by sketching the equivalent transformed section and assuming that this line would then be approximately perpendicular to the base of the structure. Using the transformed section, the value obtained for the angle between this line and the horizontal in the flow region before transformation was $72\frac{1}{2}$ degrees; the value of the corresponding angle in FIGURE 5.4 is $71\frac{1}{2}$ degrees.

FIGURES 5.5 and 5.6 are illustrations of the applicability of the computer program to flow regions consisting of irregularly shaped zones. A partial check of the solutions was made by verifying the continuity of flow condition normal to a zone boundary. The normal velocity components at eight randomly chosen points on boundaries were computed using the heads shown; the computed values on both sides of a boundary were then compared for equality. A difference of 5 to 8 percent was evident in all cases, and

was probably caused by the fact that the velocity component was an average value for some length normal to the boundary. These differences were neglected for the purpose of verifying the continuity condition.

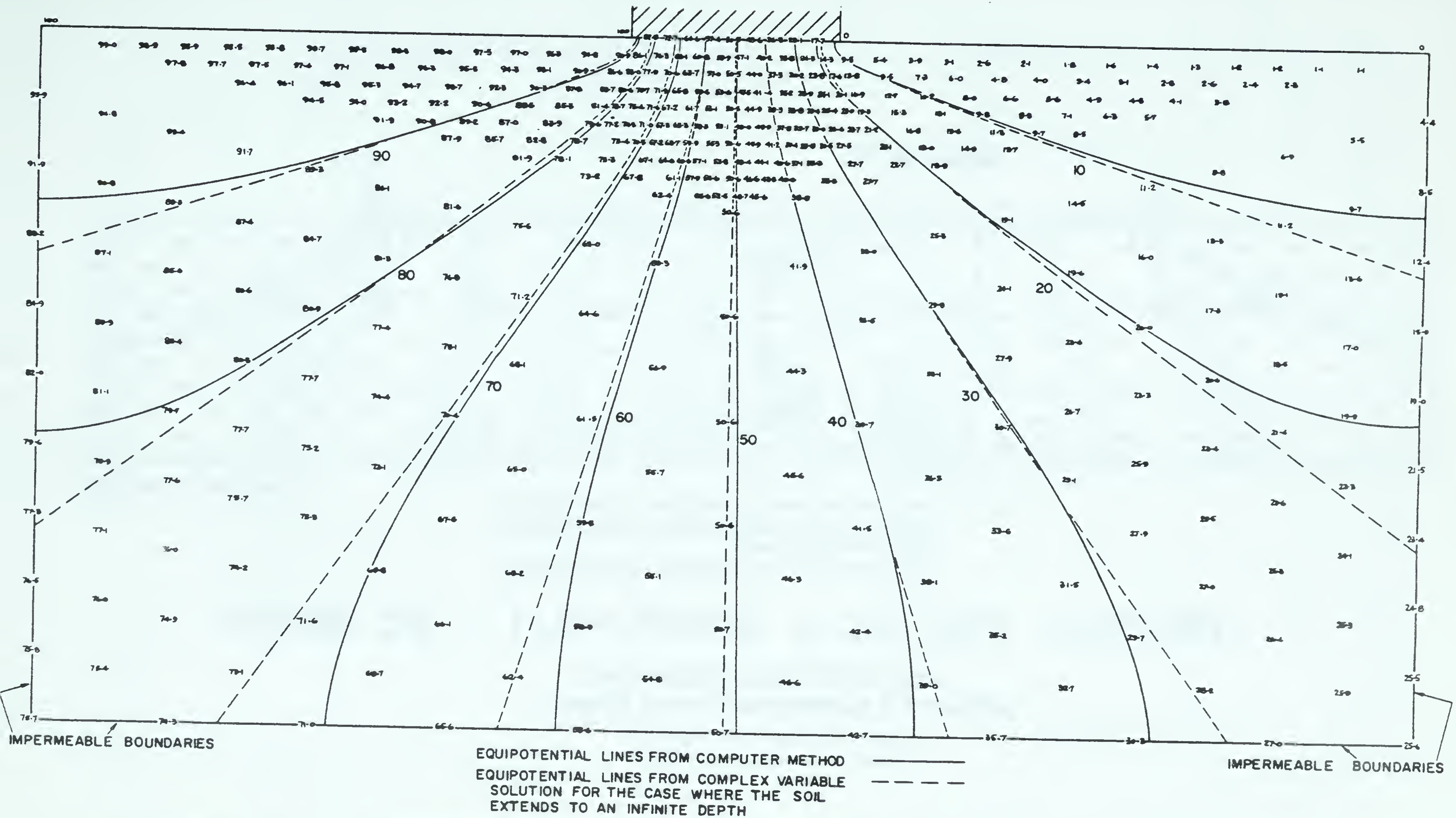
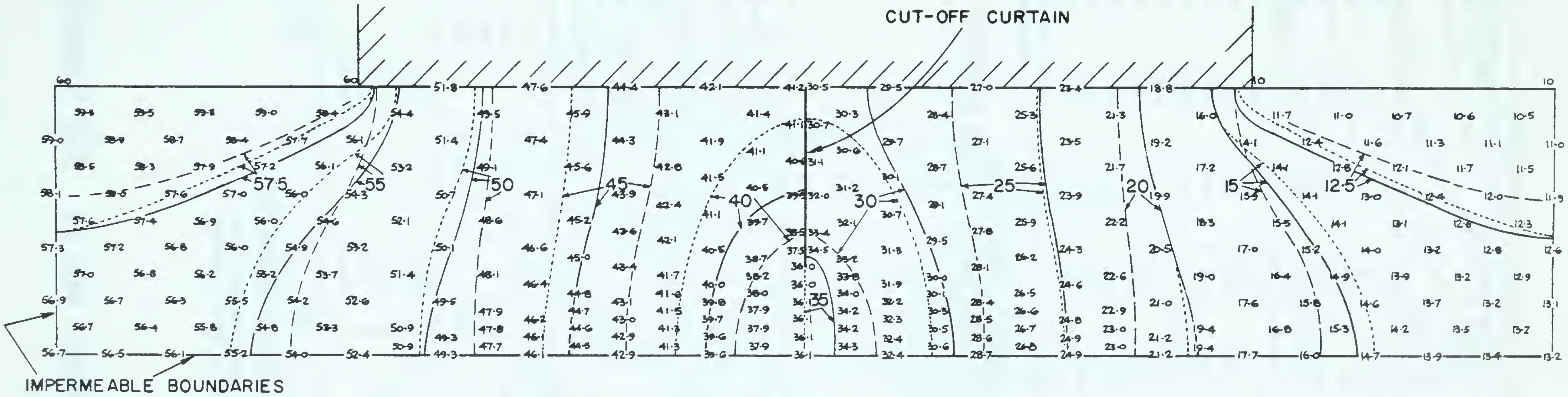


FIGURE 5·1 FLOW THROUGH A HOMOGENEOUS ISOTROPIC SOIL

NUMBER OF ITERATIONS: 137; LARGEST RESIDUAL: 0·010

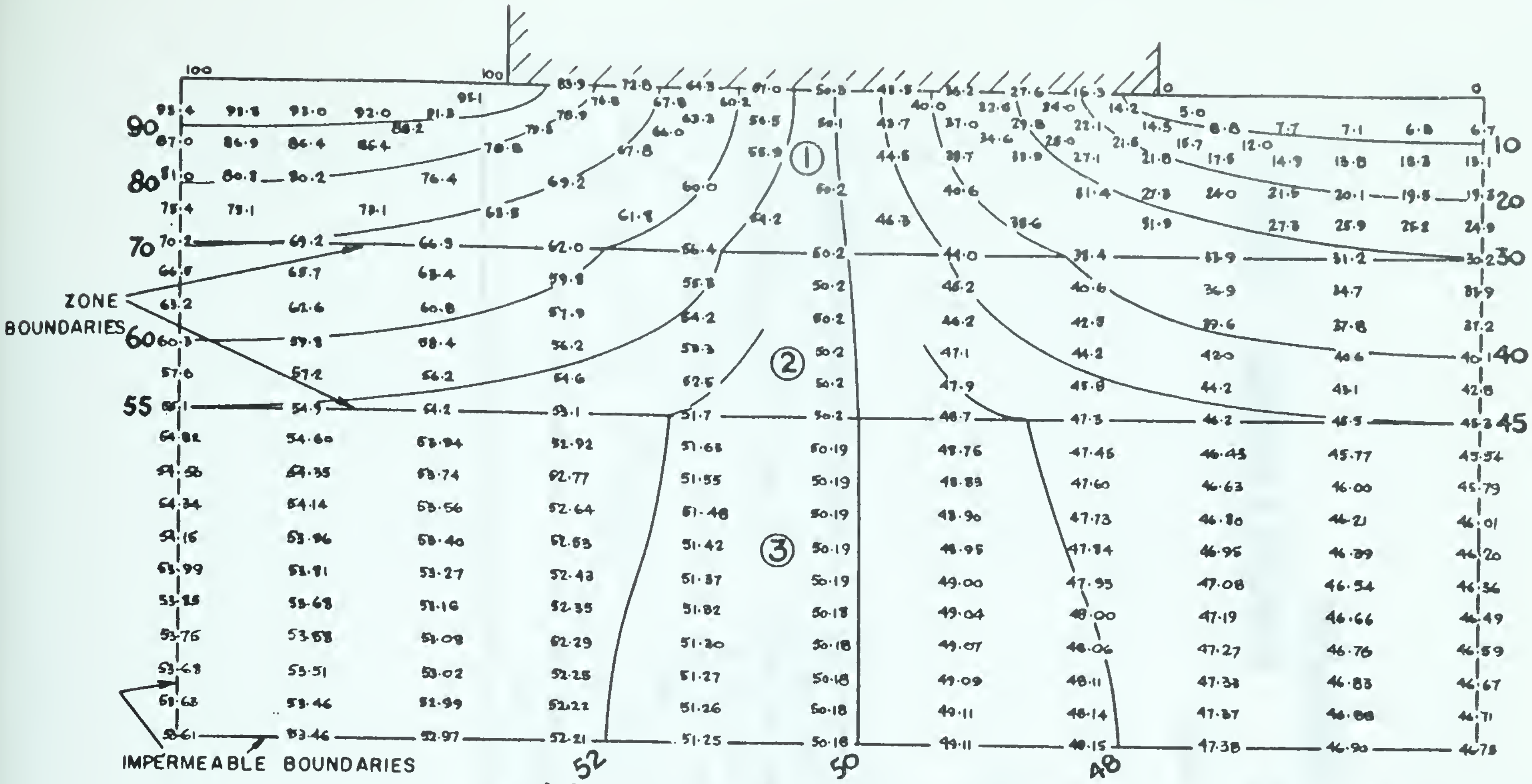


EQUIPOTENTIAL LINES FROM COMPUTER METHOD
EQUIPOTENTIAL LINES FROM RELAXATION METHOD USING A SQUARE MESH
EQUIPOTENTIAL LINES FROM ELECTRIC ANALOGUE

FIGURE 5·2 FLOW ROUND A CUT-OFF CURTAIN.

HOMOGENEOUS ISOTROPIC SOIL UNDERLAIN BY IMPERMEABLE STRATUM

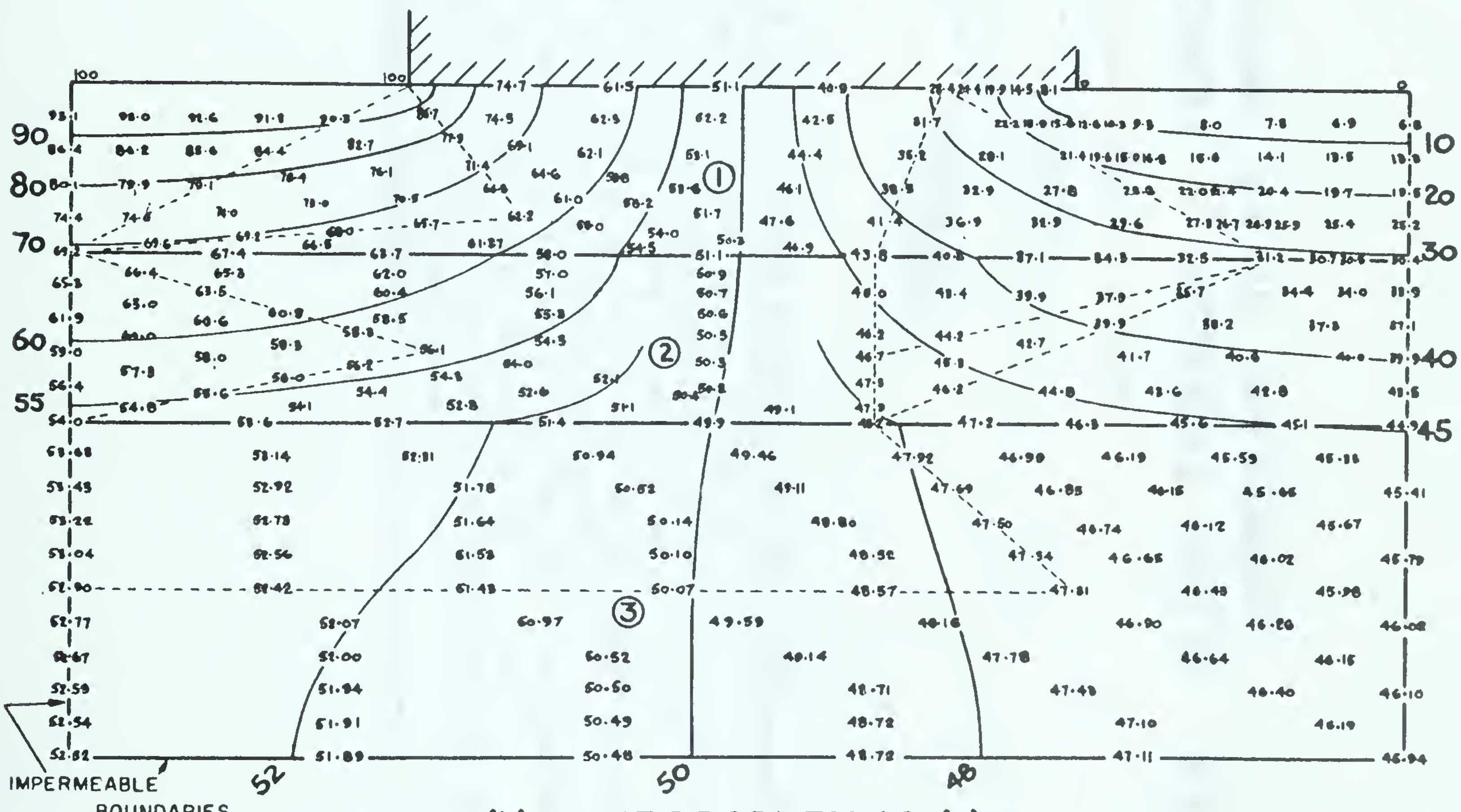
NUMBER OF ITERATIONS: 150
LARGEST RESIDUAL: 0·025



(a) FLOW UNDER STRUCTURE UNDERLAIN BY HORIZONTALLY STRATIFIED SOIL

NUMBER OF ITERATIONS: 120
LARGEST RESIDUAL: 0.010

ZONE NO	MINIMUM PERMEABILITY cm/sec x 10^{-6}	MAXIMUM PERMEABILITY cm/sec x 10^{-6}
1	6	24
2	8	32
3	64	ISOTROPIC

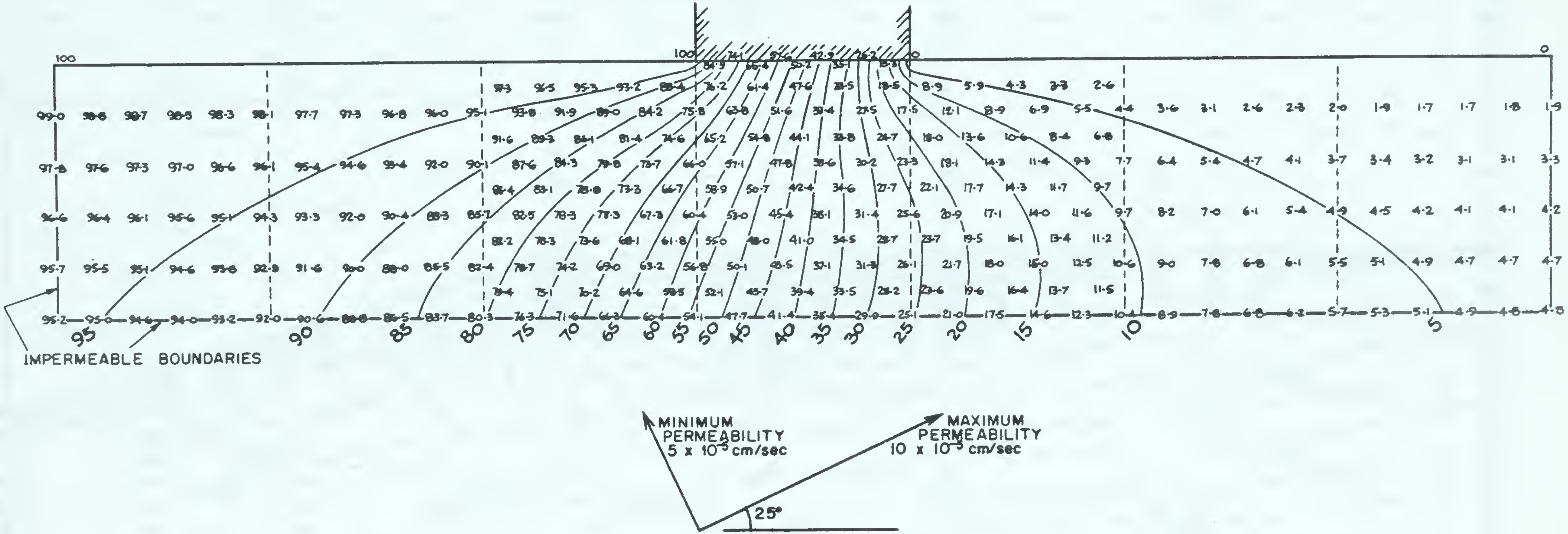


(b) SAME PROBLEM AS (a)

EXCEPT THAT EACH ZONE IS DIVIDED INTO SMALLER "ZONES" ALONG DOTTED LINES

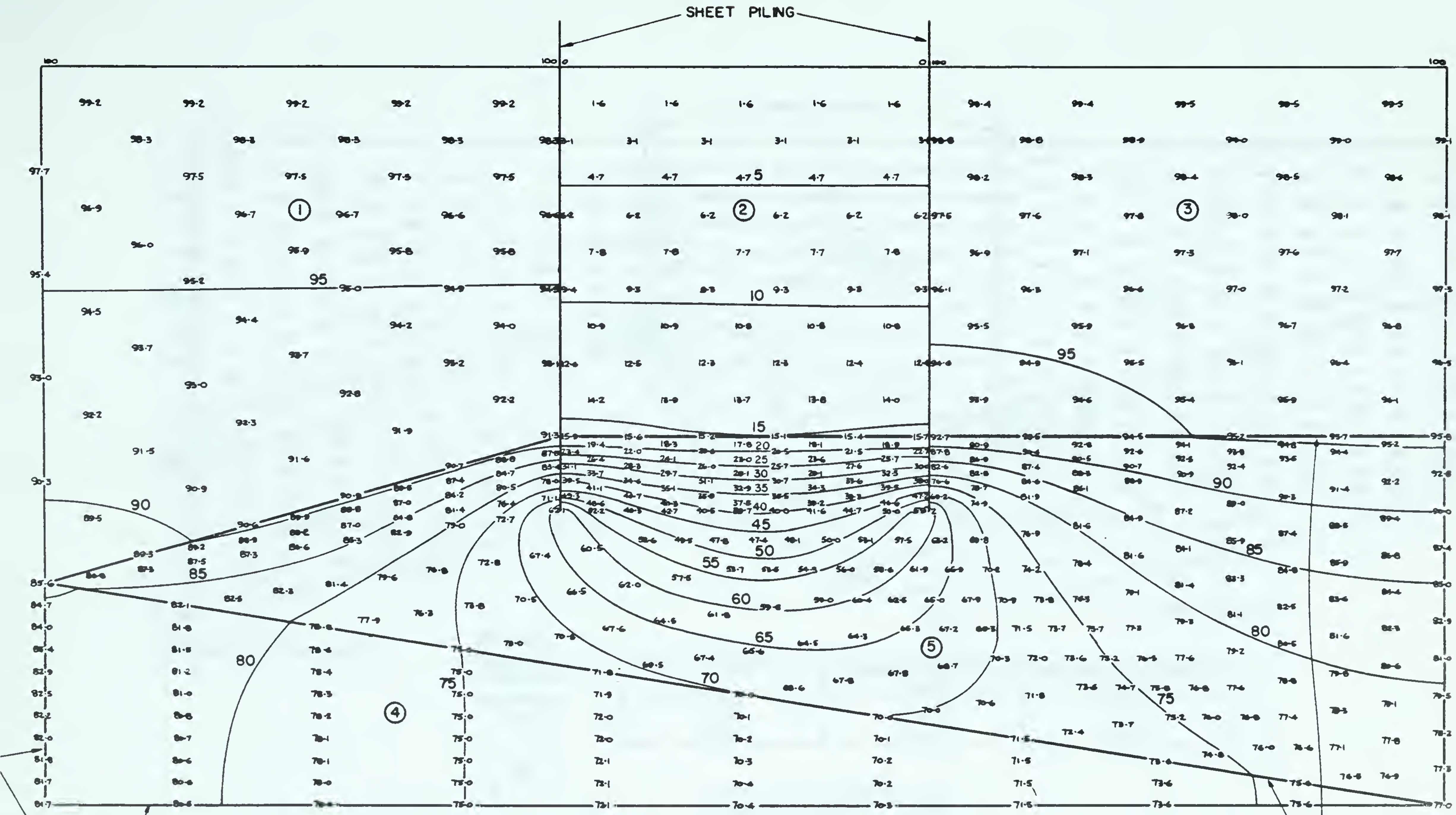
NUMBER OF ITERATIONS: 120
LARGEST RESIDUAL: 0.013

FIGURE 5.3 FLOW UNDER A STRUCTURE



NUMBER OF ITERATIONS: 200; LARGEST RESIDUAL: 0·026

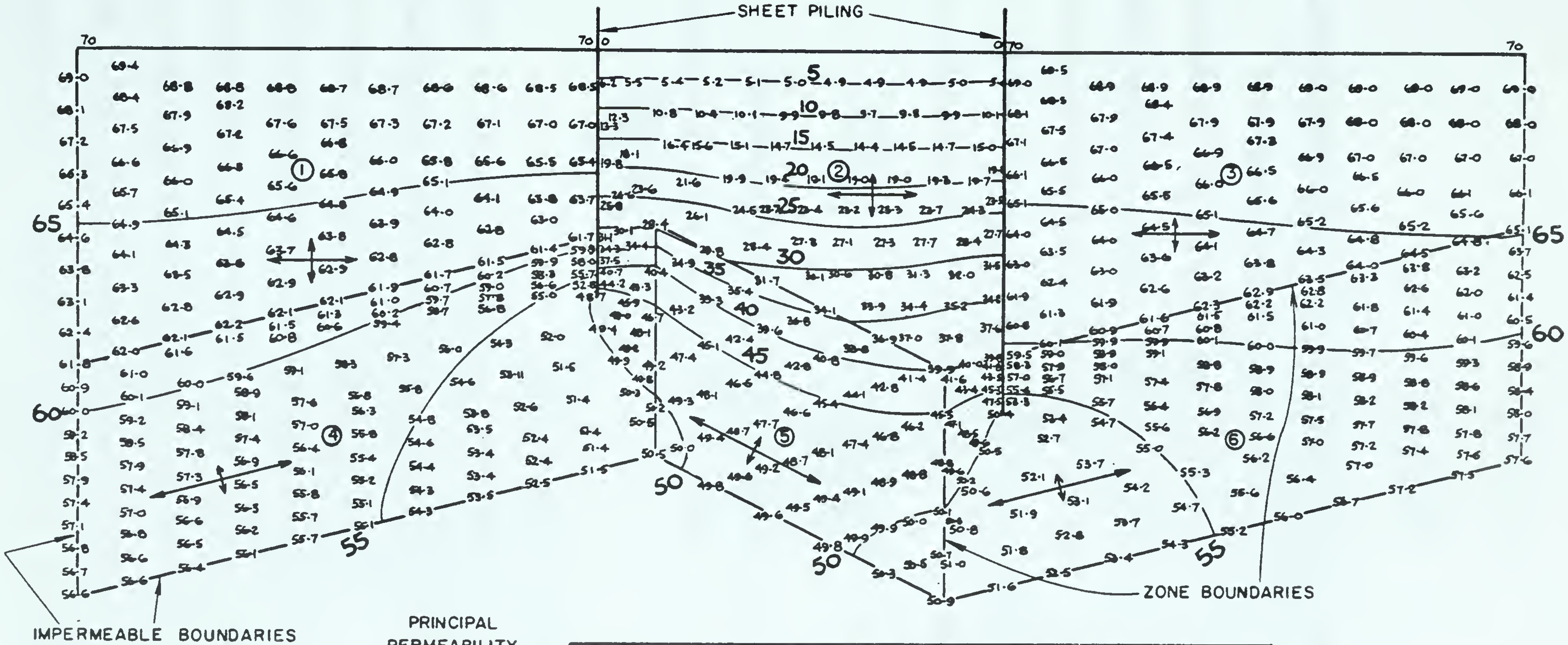
FIGURE 5·4 FLOW THROUGH A HOMOGENEOUS ANISOTROPIC SOIL UNDERLAIN BY AN IMPERMEABLE STRATUM



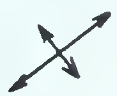
ZONE No.	DESCRIPTION	PERMEABILITY (cm/sec)
1, 2, 3, 4	ISOTROPIC	1×10^{-3}
5	HORIZONTALLY STRATIFIED	1×10^{-4} VERTICAL 2×10^{-4} HORIZONTAL

FIGURE 5·5 FLOW INTO A SHEET PILE COFFERDAM.

NUMBER OF ITERATIONS: 200; LARGEST RESIDUAL: 0·017



PRINCIPAL PERMEABILITY DIRECTIONS INDICATED THUS:



ZONE No	MINIMUM PERMEABILITY cm/sec	MAXIMUM PERMEABILITY cm/sec
1,2,3	1×10^{-5}	2×10^{-5}
4,5,6	4×10^{-6}	2×10^{-5}

NUMBER OF ITERATIONS: 200
LARGEST RESIDUAL: 0·039

FIGURE 5·6 FLOW THROUGH A FOLDED STRATUM

CHAPTER VI

DISCUSSION

6.1 The approximation in the finite difference expressions

The finite difference equations derived in CHAPTER III are approximations to the true continuity equation for flow through an elemental area in a plane. They are approximate because the expression for the second differential of head with respect to distance along an axis in terms of head values at nodes on that axis is not exact. For example, along the Ou axis this expression is stated as equation (3.14a).

$$(h)_{uu} = \frac{h_u' + h_u'' - 2h_o}{(\Delta_1 u)^2} . \tag{3.14a}$$

The terms in the Taylor's series expansions for h_u' and h_u'' which are omitted when setting up equation (3.14a) are

$$\frac{(\Delta_1 u)^2}{12} \cdot (h)_{uuuu}$$

$$+ \text{ terms of the order } (\Delta_1 u)^4, \quad (\Delta_1 u)^6, \quad \text{etc.}$$

The error in the finite difference equation for the head at a node not lying on a boundary is therefore made up of three parts, each proportional to the square of the node spacing along the three axes. Ignoring temporarily the effect of the fourth derivatives of h in any direction, the largest node spacing along any axis through a point controls the accuracy with which the head at that point can be computed. After several iterations, an error in the head at one node is manifested in the head at nodes inside an area of appreciable size. Therefore, not counting the difference in permeability

from one zone to another, or differences in the fourth derivative of head throughout the flow region, the accuracy of the head at all points is in general controlled by the largest node spacing throughout the region. This factor should be considered when the choice of node spacings is being made, but it need not be a deterrent to the choice of large and small node spacings in a flow region, if it is remembered that the accuracy in the areas where the nodes are closely spaced may be reduced by the existence of the large node spacings elsewhere. Large and small node spacings are chosen for the example shown in FIGURE 5.1, and thus the accuracy of the heads at points near the structure are probably reduced by the large node spacings near the impermeable boundaries; however, any loss of accuracy appears to be slight in terms of practical values, and is therefore outweighed by the fact that closely spaced nodes help in obtaining a better idea of the flow pattern near the structure.

The error in the finite difference equation is proportional to the fourth derivative of h. Throughout most of a flow region, this value is negligible, but there might, in some cases, be points where this value approaches infinity. Such points exist where the superficial velocity is theoretically infinite. Thus at these points finite difference expressions such as equation (3.14a) are incorrect; however, a small distance away the expressions are correct, and for this reason their use at every point in a flow region appears justified.

6.2 Over-relaxation

Because the methods of numerical approximation were devised with the intention that solutions be obtained by hand, it is frequently recommended that the estimates for the heads initially assigned to each node should be as accurate as possible, so that the amount of subsequent work is

minimised. As a result, the negative and positive residuals are randomly distributed throughout the flow region. However, when the solution is being obtained by computer, it is more convenient to assign the same initial value of head to every node except those lying on a constant potential face, even though this value may be an obviously poor guess. The number of iterations required to obtain a solution is then generally greater than if a good guess were made and in order to reduce this number, over-relaxation is used. The residual at any node is multiplied by an over-relaxation factor before adding it to the current value of the head.

The following is a hypothetical case to illustrate the effect of over-relaxation. Consider a node at which the true value of the head is 75, and the value initially assigned is 50. The values of the head after successive iterations in which over-relaxation is not used are, in order,

50, 65, 70, 73, 74, 74.5, 74.8, 74.9, 74.95 ......

The values after iterations in which over-relaxation is used are

50, 70, 76, 74.5, 75.1, 74.95 ....

It is seen that the heads converge quickly with the use of over-relaxation; and that the convergence is an oscillating process. It is important that there be a random distribution of negative and positive residuals at all heads in the flow region. Since the initial values of the head at all nodes is the same, the residuals initially are all positive in some areas and all negative in others, and it is the process of over-relaxation that disperses the signs of the residuals after sufficient iterations have been made for the convergence to oscillate. It therefore appears necessary to over-relax the residuals for two reasons, to quicken the convergence and to ensure a random distribution of negative and positive residuals.

The writer was unable to find a logical procedure for determining the over-relaxation factor. At first the value of 1.1 was chosen, but the effect on the rate of liquidation of the residuals for a number of examples was found to be negligible. The values 1.5 and 1.7 were tried, and finally 1.8 was decided upon. For a flow region similar to that shown in FIGURE 5.1, the number of iterations required to obtain a maximum residual of 0.010 was over 1000 when no over-relaxation was used; when an over-relaxation factor of 1.8 was used, the number of iterations for the same maximum residual was 109. In some examples, the oscillations of the residual at certain nodes becomes divergent with the use of over-relaxation. This situation is believed to be caused by the interaction of in-phase oscillations at adjacent nodes. The writer's method for preventing the growth of divergent oscillations was to reduce the over-relaxation factor immediately it arises. The amount by which the over-relaxation factor exceeds one is multiplied by 0.8; thus if the tendency to diverge occurs three times before the final iteration, the over-relaxation factor becomes equal to

$$1 + 0.8.\ (0.8)^3, \text{ or } 1.41.$$

Although the procedure for determining the over-relaxation factor is somewhat empirical, it appears to perform its function satisfactorily.

6.3 Methods of manipulating the residuals

The three following methods of manipulating the residuals were considered before the program was assembled in its present form:-

(a) After the head at a node is adjusted by the amount of the residual multiplied by an over-relaxation factor, the residual at that node is not required any further, and when the residual for the next node is calculated, it occupies the same storage space in the computer.

(b) The residual is computed and stored as an element of a three-dimensional array. After the residuals at all nodes are computed, the heads are adjusted.

(c) The residual is computed and stored as an element of a three-dimensional array. After all the residuals are computed, they are relaxed by adjusting the head at each node and the residuals at surrounding nodes.

In the relaxation procedure by hand using a square mesh, method (c) is used most frequently because much of the arithmetical work involves only small numbers. It is possible to see at a glance the locations of the largest residuals in the flow region and reduce these a number of times before considering other nodes. This method is not adopted in the computer solution because a large number of memory storage locations would be required for the residuals, and the determination of the adjustments to be made to the residuals at surrounding nodes would be a lengthy operation. Method (a) has the advantages that it does not require a large number of memory storage locations for the residuals, and the number of iterations is less than would occur in method (b) because the heads are adjusted after the calculation of each residual. For these reasons method (a) is adopted in the computer program.

6.4 Validity of the finite difference equation : point at the apex of a triangle

In SECTION 3.5, a finite difference equation for a point at the apex of a triangle (equation 3.43) was deduced but not proved. The data for the example of FIGURE 5.3(b) was designed so that this equation could be tested. FIGURES 5.3(a) and (b) both represent the same flow region, but

FIGURE 5.3(b) is divided into a greater number of triangles, and consequently equation (3.43) is used many more times in the analysis of FIGURE 5.3(b) than it is in FIGURE 5.3(a). The fact that the solutions from both sets of data agree closely is evidence for the soundness of the equation.

6.5 The accuracy of the solutions

The close agreement between the solutions from the computer and other methods for the examples of FIGURES 5.1 to 5.3 substantiates the validity of the computer program. The accuracy of the computer solutions could be increased by specifying a smaller tolerable residual and a greater allowable number of iterations. For practical purposes, accuracy greater than that shown in FIGURES 5.1 to 5.6 is hardly warranted and a tolerable residual equal to about 0.02 percent of the total head difference appears satisfactory for flow regions having the complexity shown.

Another factor to be considered when deciding the allowable number of iterations is the cost of computer time in relation to the accuracy obtained. For the flow region shown in FIGURE 5.1, the computation time required by the IBM 7040 computer used by the writer was 3 minutes; for the more complex flow region of FIGURE 5.6, the time was nearly 19 minutes. Until more flow regions have been analysed using the program, it is not possible to present a simple relation for estimating computer time; however, it is likely that an appreciable proportion of this time is spent in searching for the sides of triangles which lie along a common boundary.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

In existing methods of analysing the flow of water through non-homogeneous anisotropic soils, simplifying approximations in the data are necessary in order that a solution may be found within a reasonable time. Although these approximations, if made with discretion, may be small in terms of the effect on the final result, any attempt to devise a method of analysis in which these approximations are not required is justified for engineering purposes, if the procedure for solution involves no cumbersome operations and can be completed quickly. The writer concludes that such a method has been devised in this thesis.

The basic two-dimensional flow equation is developed in finite difference form in terms of three axes in a plane. The principles of numerical analysis are then used to solve this equation at nodes of tri-angular meshes superimposed on the flow region. Each triangular mesh is formed by arranging that certain axes lie along boundaries between zones of soil having different permeability, as explained in SECTION 2.4. Conventional methods of numerical analysis for flow problems use a square mesh, (Shaw and Southwell, 1941), but since the axes of a square mesh cannot usually be arranged to coincide with the boundaries of zones within a flow region, the finite difference equations for nodes situated less than one node spacing from the boundaries are awkward to set up. By adjusting the mesh configuration to suit the boundaries, as is done in the triangular mesh, the problem of

such nodes does not arise and the formation of the finite difference equation for any node in the flow region becomes systematic.

An important advantage of the method is that it is programmed for solution by computer. Due to the versatility of the triangular mesh configuration, the program can be used for the analyses of flow regions of any desired complexity, subject to the limitations imposed by the size of the computer, which are listed in SECTION 1.3.

The program is limited to cases where no phreatic line is present, and cannot therefore be used for analysing the flow of water through earth dams unless the position of the phreatic line is assumed. However, the writer believes that the program could be extended to cover cases which include a phreatic line, and his proposals in this respect are outlined in SECTION 7.2.

The validity of the computer program is substantiated by the close agreement shown in the examples of FIGURES 5.1 to 5.3 between the solutions obtained from the computer and from other methods, The general applicability of the program is illustrated by the flow regions in FIGURES 5.5 to 5.6 for which a solution by any other reasoned procedure would be impractical.

7.2 Recommendations

The computer program as presented in this thesis is by no means utterly efficient in the way the procedure for analysis is specified to the computer. This situation is largely due to the fact that the writer has concentrated on getting the method working, rather than on procedural details within the program itself. For example, the program could be improved by altering the steps involved in searching for a triangle side which adjoins a given line so that after the search has been made during the first iteration,

the number identifying the triangle side is replaced in a memory storage location in such a way that it may be obtained without a search in the second and subsequent iterations. No doubt further improvements can be made which would reduce the length of computation time, and it is suggested that some effort be made in this direction before any work is done in extending the program in any way outlined below.

Along boundaries across which water leaves the flow region it is important that the hydraulic gradient be less than that which would cause removal of the soil from the surface, i.e., piping. In this connection, it is recommended that the computer method be checked to determine if sufficient accuracy is obtained in the vicinity of these boundaries. In a strict sense, computation of the hydraulic gradient requires that the direction of flow be known; this direction is perpendicular to the equipotential line at any point in an isotropic soil, but this rule does not exist in an anisotropic soil. The direction of flow can be obtained from a knowledge of the flow lines, and it is suggested that these may be determined by computing the velocity components in the principal permeability directions at nodes situated where the flow lines are required.

A natural extension of this thesis would be an adaptation of the program to cover flow regions which have a phreatic line as a boundary. The writer suggests that this adaptation could be made by initially estimating the location of the phreatic line, in much the same way that an initial guess was made for the head at each node, and then treating that line as if it were an impermeable boundary; if the line is curved it is approximated to a number of straight lines. After a sufficient number of iterations has been made to cause residuals at nodes to become tolerably small, the trial phreatic line is

inspected for the condition that the change in total head between any two points along its length is equal to the difference in vertical elevation between those points. The location of the phreatic line is altered until this condition is satisfied within tolerable limits. With each alteration the mesh configuration is adjusted to suit the new alignment of the phreatic line and new values for the heads at nodes are determined. In effect, the procedure for analysing flow regions which do not include a phreatic line is repeated until the location of the phreatic line, temporarily regarded as an impermeable boundary, is determined correctly through successive approximations. The presence of a seepage surface may be regarded for the purposes of the analysis as a boundary between two zones, one of which is infinitely permeable and equations (3.46) to (3.49) adjusted to suit this condition.

It is recommended that the concept of a triangular mesh be used as a basis for investigating steady-state three-dimensional flow conditions in non-homogeneous soils. Each zone could be approximated to a solid figure whose sides consist of irregular triangles; the zone can then be divided into a number of tetrahedra. The finite difference equation for a node may be expressed in terms of the six axes which are parallel to the sides of each tetrahedron. The positions of nodes can then be classified under four headings, as follows: (a) inside a tetrahedron, (b) on a side of a tetrahedron, (c) on an edge of a tetrahedron, or (d) at an apex of a tetrahedron; the procedure for solution is then likely to be similar to that used for two-dimensional flow.

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Appendix A

CONTINUITY EQUATION :

DERIVATION

APPENDIX A

CONTINUITY EQUATION : DERIVATION

Consider an arbitary point O in an anisotropic homogeneous soil in which flow occurs in two-dimensions only, defined by the plane of Ox and Oy. Let Ox and Oy be the directions of principal permeabilities, and $v_{x,y}$ the components of (superficial) velocity at O, (FIGURE A1).

The velocity components at P on the circumference of the elemental circle centred on O, radius Δs, are equal to the velocity components at O plus the rate of change of the velocity components in the s- direction multiplied by Δs, i.e.,[1]

velocity component at P in x- direction = $v_x + (v_x)_s . \Delta s$, and

velocity component at P in y- direction = $v_y + (v_y)_s . \Delta s$.

Therefore the flow q across PP' is equal to the radial velocity component, multiplied by the arc length PP'. Resolving the above velocity components in the radial direction,

---

[1] Partial derivatives are denoted herein as follows, e.g:

$(h)_u$, which represents $\frac{\partial h}{\partial u}$, or the partial derivative of h with respect to u, and

$(h)_{uv}$, representing $\frac{\partial^2 h}{\partial u . \partial v}$, or the second partial derivative of h with respect to u and v.

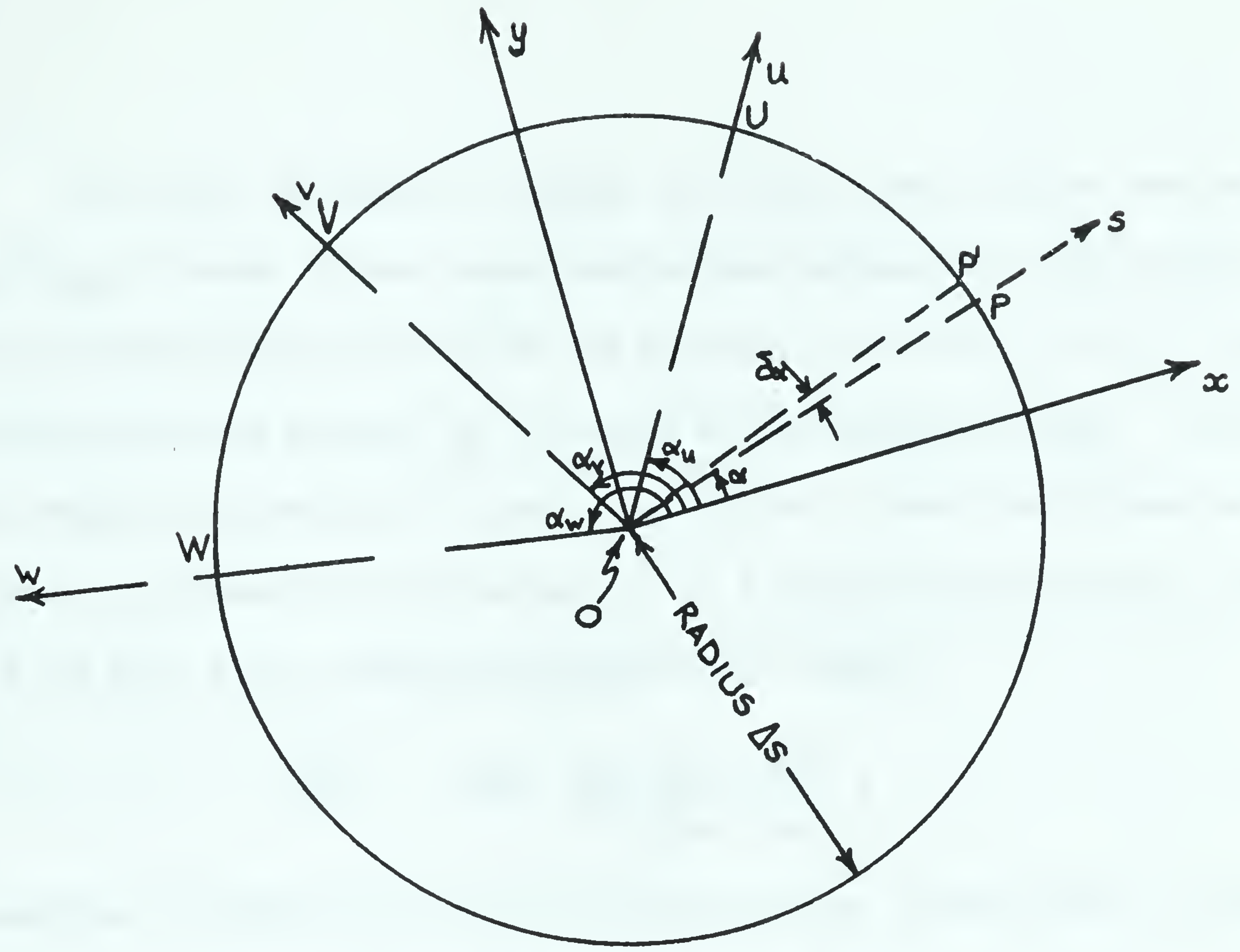

FIGURE A1. Elemental circle.

$$q = \{(v_x + (v_x)_s \cdot \Delta s)\cos\alpha \quad + \quad (v_y + (v_y)_s \cdot \Delta s)\sin\alpha\}\,\Delta s \cdot \delta\alpha$$

$$\left.\begin{aligned} &= \{v_x \cos\alpha \;+\; v_y \sin\alpha\}\,\Delta s.\ \delta\alpha \\ &\qquad \{(v_x)_s \cos\alpha \;+\; (v_y)_s \sin\alpha\}.\,(\Delta s)^2.\ \delta\alpha. \end{aligned}\right\} \quad (3.1)$$

But, by Darcy's law, $v_x = -k_x(h)_x$ and $v_y = -k_y(h)_y$ where $k_{x,y}$ are the permeabilities in the x- and y- directions respectively. Therefore, differentiating the expressions for v_x and v_y,

$$(v_x)_s = -k_x(h)_{xs}, \tag{3.2a}$$

$$(v_y)_s = -k_y(h)_{ys}. \tag{3.2b}$$

Substituting equations (3.2a) and (3.2b) in equation (3.1),

$$\begin{aligned} q = &\{v_x \cos\alpha \;+\; v_y \sin\alpha\}.\ \Delta s \cdot \delta\alpha \\ &- \{k_x(h)_{xs}\cos\alpha \;+\; k_y(h)_{ys}\sin\alpha\}.\,(\Delta s)^2.\ \delta\alpha. \end{aligned} \tag{3.3}$$

It is now necessary to express the mixed second partial derivatives $(h)_{xs}$, $(h)_{ys}$ in terms of the second partial derivatives of h with respect to distances measured along the Ou, Ov and Ow axes, (i.e., $(h)_{uu}$, $(h)_{vv}$, $(h)_{ww}$), where these axes are parallel to the sides of the triangular mesh. Two axes (albeit skew) are sufficient to define a point in a plane; the Ou and Ow axes are arbitarily chosen for this purpose. For a change from the point O to the point P (by Δs), the u- dimension changes by Δu, where

$$\Delta u = \Delta s . \frac{\sin(\alpha_W - \alpha)}{\sin(\alpha_W - \alpha_u)} ;$$

this equation is obtained by use of the sine rule on triangle OPW''' (FIGURE A2). In the limit, Δu becomes ∂u and Δs becomes ∂s, so that

$$(u)_s = \frac{\sin(\alpha_W - \alpha)}{\sin(\alpha_W - \alpha_u)} , \qquad (3.4a)$$

Similarly, it can be shown that

$$(w)_s = - \frac{\sin(\alpha_u - \alpha)}{\sin(\alpha_W - \alpha_u)} , \qquad (3.4b)$$

and

$$(u)_x = \frac{\sin \alpha_W}{\sin(\alpha_W - \alpha_u)} , \quad (3.4c), \qquad (w)_x = - \frac{\sin \alpha_u}{\sin(\alpha_W - \alpha_u)} , (3.4d),$$

$$(u)_y = - \frac{\cos \alpha_W}{\sin(\alpha_W - \alpha_u)} , \quad (3.4e), \qquad (w)_y = \frac{\cos \alpha_u}{\sin(\alpha_W - \alpha_u)} , (3.4f),$$

$$(u)_v = \frac{\sin(\alpha_W - \alpha_v)}{\sin(\alpha_W - \alpha_u)} , \quad (3.4g), \qquad (w)_v = - \frac{\sin(\alpha_u - \alpha_v)}{\sin(\alpha_W - \alpha_u)} , (3.4h)$$

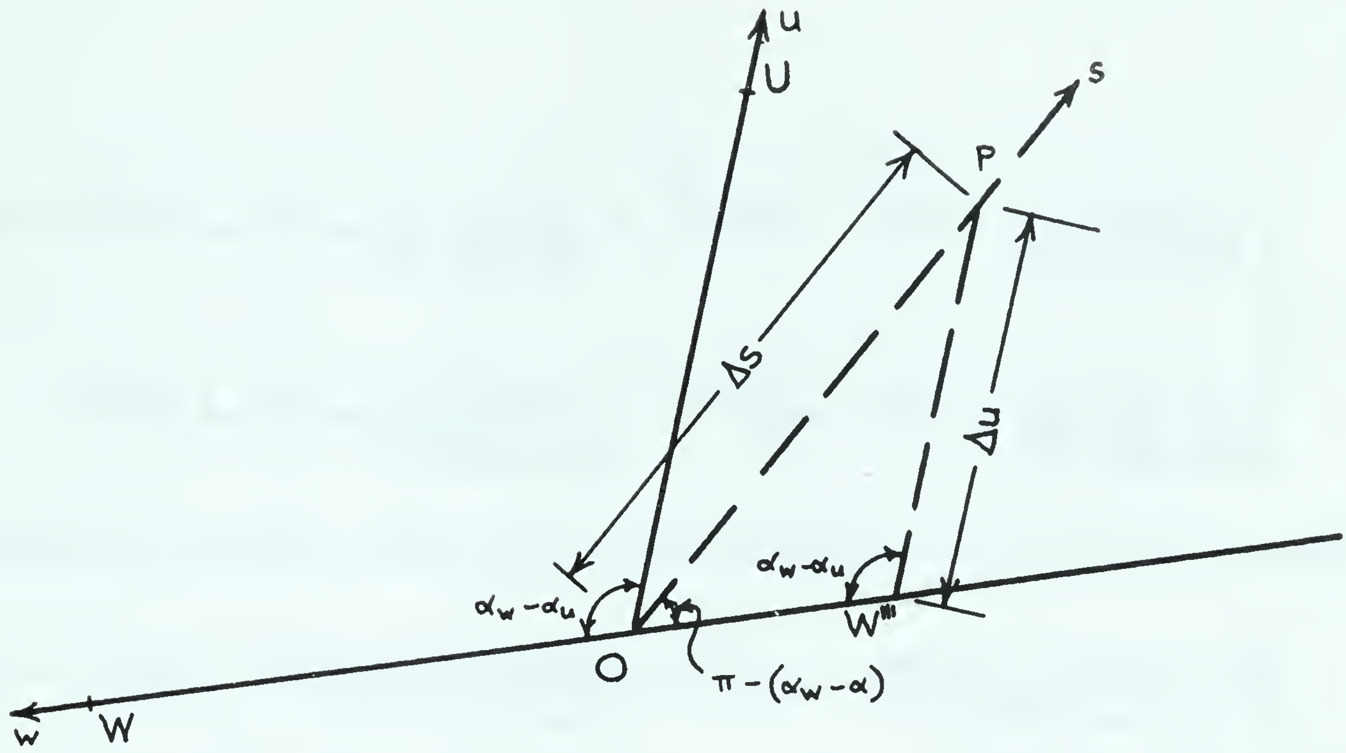

FIGURE A2.

Continuing to work in terms of the Ou and Ow axes, any distance along Ox, Oy, Os or Ov may be regarded as a function of u and w only; therefore

$$(h)_s = (h)_u \cdot (u)_s + (h)_w \cdot (w)_s . \tag{3.5}$$

Substituting equations (3.4a) and (3.4b) in equation (3.5),

$$(h)_s = (h)_u \cdot \frac{\sin(\alpha_w - \alpha)}{\sin(\alpha_w - \alpha_u)} - (h)_w \cdot \frac{\sin(\alpha_u - \alpha)}{\sin(\alpha_w - \alpha_u)} . \tag{3.6}$$

Differentiating both sides of equation (3.6) with respect to x,

$$((h)_s)_x = \left((h)_u \cdot \frac{\sin(\alpha_w - \alpha)}{\sin(\alpha_w - \alpha_u)}\right)_x - \left((h)_w \cdot \frac{\sin(\alpha_u - \alpha)}{\sin(\alpha_w - \alpha_u)}\right)_x ,$$

$$\left.\begin{aligned}(h)_{xs} = {} & ((h)_u)_u \cdot (u)_x \cdot \frac{\sin(\alpha_w-\alpha)}{\sin(\alpha_w-\alpha_u)} + ((h)_u)_w \cdot (w)_x \cdot \frac{\sin(\alpha_w-\alpha)}{\sin(\alpha_w-\alpha_u)} \\ & -((h)_w)_u \cdot (u)_x \cdot \frac{\sin(\alpha_u-\alpha)}{\sin(\alpha_w-\alpha_u)} - ((h)_w)_w \cdot (w)_x \cdot \frac{\sin(\alpha_u-\alpha)}{\sin(\alpha_w-\alpha_u)}\end{aligned}\right\} (3.7)$$

Substituting equations (3.4c) and (3.4d) in equation (3.7), and rearranging,

$$\left.\begin{aligned}(h)_{xs} = {} & (h)_{uu} \cdot \frac{\sin\alpha_w \cdot \sin(\alpha_w-\alpha)}{\sin^2(\alpha_w-\alpha_u)} + (h)_{ww} \cdot \frac{\sin\alpha_u \cdot \sin(\alpha_u-\alpha)}{\sin^2(\alpha_w-\alpha_u)} \\ & -(h)_{uw} \cdot \frac{\sin\alpha_u \cdot \sin(\alpha_w-\alpha) + \sin\alpha_w \cdot \sin(\alpha_u-\alpha)}{\sin^2(\alpha_w-\alpha_u)}\end{aligned}\right\} (3.8a)$$

Similarly, using equations (3.4e) and (3.4f), it can be shown that

$$\left.\begin{aligned}(h)_{ys} = {} & -(h)_{uu} \cdot \frac{\cos\alpha_w \cdot \sin(\alpha_w-\alpha)}{\sin^2(\alpha_w-\alpha_u)} - (h)_{ww} \cdot \frac{\cos\alpha_u \cdot \sin(\alpha_u-\alpha)}{\sin^2(\alpha_w-\alpha_u)} \\ & +(h)_{uw} \cdot \frac{\cos\alpha_u \cdot \sin(\alpha_w-\alpha) + \cos\alpha_w \cdot \sin(\alpha_u-\alpha)}{\sin^2(\alpha_w-\alpha_u)} .\end{aligned}\right\} (3.8b)$$

Using equations (3.4g) and (3.4h), it can be shown that

$$\left.\begin{aligned}(h)_{vv} = {} & (h)_{uu} \cdot \frac{\sin^2(\alpha_w-\alpha_v)}{\sin^2(\alpha_w-\alpha_u)} + (h)_{ww} \cdot \frac{\sin^2(\alpha_u-\alpha_v)}{\sin^2(\alpha_w-\alpha_u)} \\ & -2(h)_{uw} \cdot \frac{\sin(\alpha_w-\alpha_v) \cdot \sin(\alpha_u-\alpha_v)}{\sin^2(\alpha_w-\alpha_u)} .\end{aligned}\right\} (3.8c)$$

Equation (3.8c) may be used to eliminate $(h)_{uw}$ from equations (3.8a) and (3.8b), resulting in

$$\left.\begin{aligned}(h)_{xs} &= (h)_{uu}\cdot\frac{\sin\alpha_v.\sin(\alpha-\alpha_w)+\sin\alpha_w.\sin(\alpha-\alpha_v)}{2\sin(\alpha_v-\alpha_u).\sin(\alpha_u-\alpha_w)}\\ &+(h)_{vv}\cdot\frac{\sin\alpha_w.\sin(\alpha-\alpha_u)+\sin\alpha_u.\sin(\alpha-\alpha_w)}{2\sin(\alpha_w-\alpha_v).\sin(\alpha_v-\alpha_u)}\\ &+(h)_{ww}\cdot\frac{\sin\alpha_u.\sin(\alpha-\alpha_v)+\sin\alpha_v.\sin(\alpha-\alpha_u)}{2\sin(\alpha_u-\alpha_w).\sin(\alpha_w-\alpha_v)},\end{aligned}\right\} \quad (3.9a)$$

and

$$\left.\begin{aligned}(h)_{ys} &= -(h)_{uu}\cdot\frac{\cos\alpha_v.\sin(\alpha-\alpha_w)+\cos\alpha_w.\sin(\alpha-\alpha_v)}{2\sin(\alpha_v-\alpha_u).\sin(\alpha_u-\alpha_w)}\\ &-(h)_{vv}\cdot\frac{\cos\alpha_w.\sin(\alpha-\alpha_u)+\cos\alpha_u.\sin(\alpha-\alpha_w)}{2.\sin(\alpha_w-\alpha_v).\sin(\alpha_v-\alpha_u)}\\ &-(h)_{ww}\cdot\frac{\cos\alpha_u.\sin(\alpha-\alpha_v)+\cos\alpha_v.\sin(\alpha-\alpha_u)}{2\sin(\alpha_u-\alpha_w).\sin(\alpha_w-\alpha_v)}.\end{aligned}\right\} \quad (3.9b)$$

Equations (3.9a) and (3.9b) may be written as follows:

$$(h)_{xs} = \sum_{M=u,v,w}(h)_{MM}\cdot\frac{\sin\alpha_N.\sin(\alpha-\alpha_P)+\sin\alpha_P.\sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M).\sin(\alpha_M-\alpha_P)}, \quad (3.9c)$$

and

$$(h)_{ys} = \sum_{M=u,v,w}(h)_{MM}\cdot\frac{\cos\alpha_N.\sin(\alpha-\alpha_P)+\cos\alpha_P.\sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M).\sin(\alpha_M-\alpha_P)}, \quad (3.9d)$$

where M, N and P are, respectively, u, v, w,
and v, w, u,
and w, u, v.

Substituting equations (3.9c) and (3.9d) in equation (3.3), the flow q across PP' is given by

$$q = \left\{ v_x \cos\alpha + v_y \sin\alpha \right\} . \Delta s . \delta\alpha$$

$$- \left\{ k_x . \cos\alpha \sum (h)_{MM} . \frac{\sin\alpha_N . \sin(\alpha-\alpha_P) + \sin\alpha_P . \sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M) . \sin(\alpha_M-\alpha_P)} \right\} . (\Delta s)^2 . \delta\alpha$$

$$+ \left\{ k_y . \sin\alpha \sum (h)_{MM} . \frac{\cos\alpha_N . \sin(\alpha-\alpha_P) + \cos\alpha_P . \sin(\alpha-\alpha_N)}{2\sin(\alpha_N-\alpha_M) . \sin(\alpha_M-\alpha_P)} \right\} . (\Delta s)^2 . \delta\alpha$$

$$\left. \begin{aligned} &= \left\{ v_x \cos\alpha + v_y \sin\alpha \right\} . \Delta s . \delta\alpha \\ &- \left\{ \sum (h)_{MM} . \left[\frac{(k_x+k_y)\sin(\alpha_N+\alpha_P)\cos\alpha\sin\alpha - 2k_x \sin\alpha_N \sin\alpha_P \cos^2\alpha - 2k_y \cos\alpha_N \cos\alpha_P \sin^2\alpha}{2\sin(\alpha_N-\alpha_M) . \sin(\alpha_M-\alpha_P)} \right] \right\} (\Delta s)^2 . \delta\alpha \end{aligned} \right\} \quad (3.10)$$

Continuity requires that the net outflow from the circle is zero. This condition may be stated algebraicly as

$$\int_0^{2\pi} q \, d\alpha = 0 .$$

If O is a point inside a homogeneous soil, then k_x and k_y are constant values. Thus, integrating equation (3.10) between O and 2π,

$$\int_0^{2\pi} q \, d\alpha = \int_0^{2\pi} \left\{ v_x \cos\alpha + v_y \sin\alpha \right\} . \Delta s . d\alpha$$

$$- \int_0^{2\pi} \left\{ \sum (h)_{MM} \left[\frac{(k_x+k_y) . \sin(\alpha_N+\alpha_P)\cos\alpha\sin\alpha - 2k_x \sin\alpha_N \sin\alpha_P \cos^2\alpha - 2k_y \cos\alpha_N \cos\alpha_P \sin^2\alpha}{2\sin(\alpha_N-\alpha_M .) . \sin(\alpha_M-\alpha_P)} \right] \right\} (\Delta s)^2 . d\alpha$$

$$= \pi (\Delta s)^2 . \sum (h)_{MM} . \frac{k_x \sin\alpha_N \sin\alpha_P + k_y \cos\alpha_N \cos\alpha_P}{\sin(\alpha_N-\alpha_M) . \sin(\alpha_M-\alpha_P)} .$$

Therefore, for continuity,

$$\sum_{M=u,v,w} (h)_{MM} . \frac{k_x \sin\alpha_N . \sin\alpha_P + k_y \cos\alpha_N . \cos\alpha_P}{\sin(\alpha_N-\alpha_M) . \sin(\alpha_M-\alpha_P) .} = 0 . \quad (3.11)$$

Equation (3.11) is the continuity equation of flow, in terms of three axes in a plane.

Appendix B

FINITE DIFFERENCE EQUATIONS :

DERIVATION

APPENDIX B

FINITE DIFFERENCE EQUATIONS : DERIVATION

Point inside a homogeneous soil

Consider a point O at a node in a triangular mesh superimposed on a homogeneous anisotropic soil. The Ou, Ov and Ow axes are at angles α_u, α_v and α_w measured in a counter-clockwise sense from the direction Ox of minimum permeability. The angle between Ox and some datum direction (arbitarily taken as horizontal) is α_a. The points U', V' and W' are one mesh length from O in a positive direction along the Ou, Ov and Ow axes respectively; U", V" and W" are one mesh length in a negative direction along the same axes. Let $\Delta_1 u$, $\Delta_1 v$, $\Delta_1 w$ be the mesh lengths along the Ou, Ov and Ow axes respectively, (FIGURE B1).

Denoting the head at O by h_o, at U' by h'_u, and at U" by h''_u, the variation of head along Ou may be expressed by Taylor's theorem as follows:

$$h'_u = h_o + \Delta_1 u.(h)_u + \frac{(\Delta_1 u)^2}{2!}.(h)_{uu} + \frac{(\Delta_1 u)^3}{3!}.(h)_{uuu} + \frac{(\Delta_1 u)^4}{4!}.(h)_{uuuu} + \ldots, \quad (3.12a)$$

and

$$h''_u = h_o - \Delta_1 u.(h)_u + \frac{(\Delta_1 u)^2}{2!}.(h)_{uu} - \frac{(\Delta_1 u)^3}{3!}.(h)_{uuu} + \frac{(\Delta_1 u)^4}{4!}.(h)_{uuuu} + \ldots, \quad (3.12b)$$

Adding equations (3.12a) and (3.12b) gives

$$(h)_{uu} = \frac{h'_u + h''_u - 2h_o}{(\Delta_1 u)^2} - \frac{(\Delta_1 u)^2}{12}.(h)_{uuuu} - \ldots\ldots \quad (3.13)$$

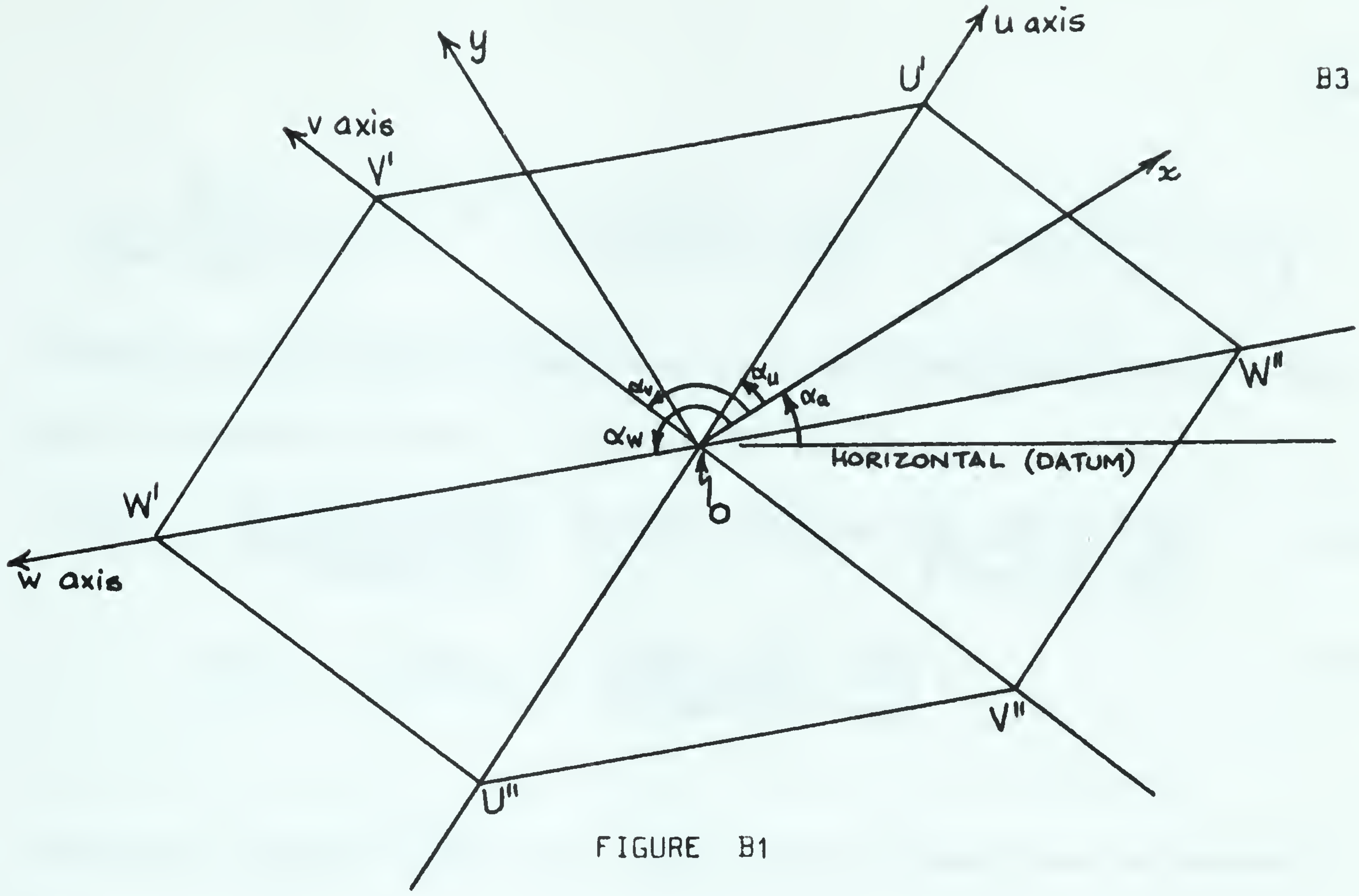

FIGURE B1

By choosing a mesh size which is small, terms containing $\Delta_1 u$ raised to the power of 2 or greater may be neglected, and the second derivative of head with respect to distance along Ou is given approximately by

$$(h)_{uu} = \frac{h'_u + h''_u - 2h_o}{(\Delta_1 u)^2} . \qquad (3.14a)$$

Similarly, with respect to distances along Ov and Ow,

$$(h)_{vv} = \frac{h'_v + h''_v - 2h_o}{(\Delta_1 v)^2} , \qquad (3.14b)$$

and

$$(h)_{ww} = \frac{h'_w + h''_w - 2h_o}{(\Delta_1 w)^2} . \qquad (3.14c)$$

Now, referring to FIGURE B1, the length U'V' = length OW' = $\Delta_1 w$. Therefore, using the sine rule on triangle OU'V',

$$\frac{\Delta_1 u}{\sin(\alpha_W - \alpha_V)} = \frac{\Delta_1 v}{\sin(\alpha_W - \alpha_u)} = \frac{\Delta_1 w}{\sin(\alpha_V - \alpha_u)}$$

Denoting each of the above equalities by F, and substituting for $\Delta_1 u$, $\Delta_1 v$, $\Delta_1 w$ in equations (3.14a), (3.14b) and (3.14c) gives

$$(h)_{uu} = \frac{h'_u + h''_u - 2h_o}{F^2 \sin^2(\alpha_W - \alpha_V)} \quad (3.15a), \quad (h)_{VV} = \frac{h'_V + h''_V - 2h_o}{F^2 \sin^2(\alpha_W - \alpha_u)} \quad (3.15b),$$

and

$$(h)_{WW} = \frac{h'_W + h''_W - 2h_o}{F^2 \sin^2(\alpha_V - \alpha_u)}. \quad (3.15c)$$

Equations (3.15a), (3.15b) and (3.15c) can be expressed more conveniently as

$$(h)_{MM} = \frac{h'_M + h''_M - 2h_o}{F^2 \sin^2(\alpha_P - \alpha_N)}, \quad (3.16)$$

where M, N and P have the same meanings as were used in equations (3.9c) and (3.9d).

The finite difference expression of equation (3.16) can be substituted in equation (3.11) to give an equation relating h_o to h'_u, h''_u, h'_V, h''_V, h'_W and h''_W, as follows:

$$\sum_{M=u,v,w} (h'_M + h''_M - 2h_o) \cdot \frac{k_x \sin\alpha_N \cdot \sin\alpha_P + k_y \cos\alpha_N \cdot \cos\alpha_P}{F^2 \sin^2(\alpha_P - \alpha_N) \sin(\alpha_N - \alpha_M) \sin(\alpha_M - \alpha_P)} = 0.$$

Each term in the summation may be multiplied by the common factor $[\, F^2 \sin(\alpha_P - \alpha_N) \sin(\alpha_N - \alpha_M) \sin(\alpha_M - \alpha_P) \,]$ to give

$$\sum_{M=u,v,w} (h'_M + h''_M - 2h_o) \cdot \frac{k_x \sin\alpha_N \cdot \sin\alpha_P + k_y \cos\alpha_N \cdot \cos\alpha_P}{\sin(\alpha_N - \alpha_P)} = 0. \quad (3.17)$$

Denoting the coefficients C by

$$C_u = \frac{k_x \sin\alpha_v \sin\alpha_w + k_y \cos\alpha_v \cos\alpha_w}{\sin(\alpha_v - \alpha_w)},$$

$$C_v = \frac{k_x \sin\alpha_w \sin\alpha_u + k_y \cos\alpha_w \cos\alpha_u}{\sin(\alpha_w - \alpha_u)},$$

$$C_w = \frac{k_x \sin\alpha_u \sin\alpha_v + k_y \cos\alpha_u \cos\alpha_v}{\sin(\alpha_u - \alpha_v)}$$

equation (3.17) can be written as

$$C_u.h'_u + C_v.h'_v + C_w.h'_w + C_u.h''_u + C_v.h''_v + C_w.h''_w - 2(C_u + C_v + C_w)h_o = 0. \qquad (3.18)$$

Equation (3.18) is the finite difference equation for the head at a point inside a homogeneous soil.

It is more convenient to express the directions of the Ou, Ov and Ow axes as the angle between each and the datum (horizontal) direction. If these angles are denoted by α'_u, α'_v and α'_w respectively, the coefficients C_u, C_v and C_w in equation (3.18) become

$$C_u = \frac{k_x \sin(\alpha'_v - \alpha_a)\sin(\alpha'_w - \alpha_a) + k_y \cos(\alpha'_v - \alpha_a)\cos(\alpha'_w - \alpha_a)}{\sin(\alpha'_v - \alpha'_w)}, \qquad (3.19a)$$

$$C_v = \frac{k_x \sin(\alpha'_w - \alpha_a)\sin(\alpha'_u - \alpha_a) + k_y \cos(\alpha'_w - \alpha_a)\cos(\alpha'_u - \alpha_a)}{\sin(\alpha'_w - \alpha'_u)}, \qquad (3.19b)$$

$$C_w = \frac{k_x \sin(\alpha'_u - \alpha_a)\sin(\alpha'_v - \alpha_a) + k_y \cos(\alpha'_u - \alpha_a)\cos(\alpha'_v - \alpha_a)}{\sin(\alpha'_u - \alpha'_v)}. \qquad (3.19c)$$

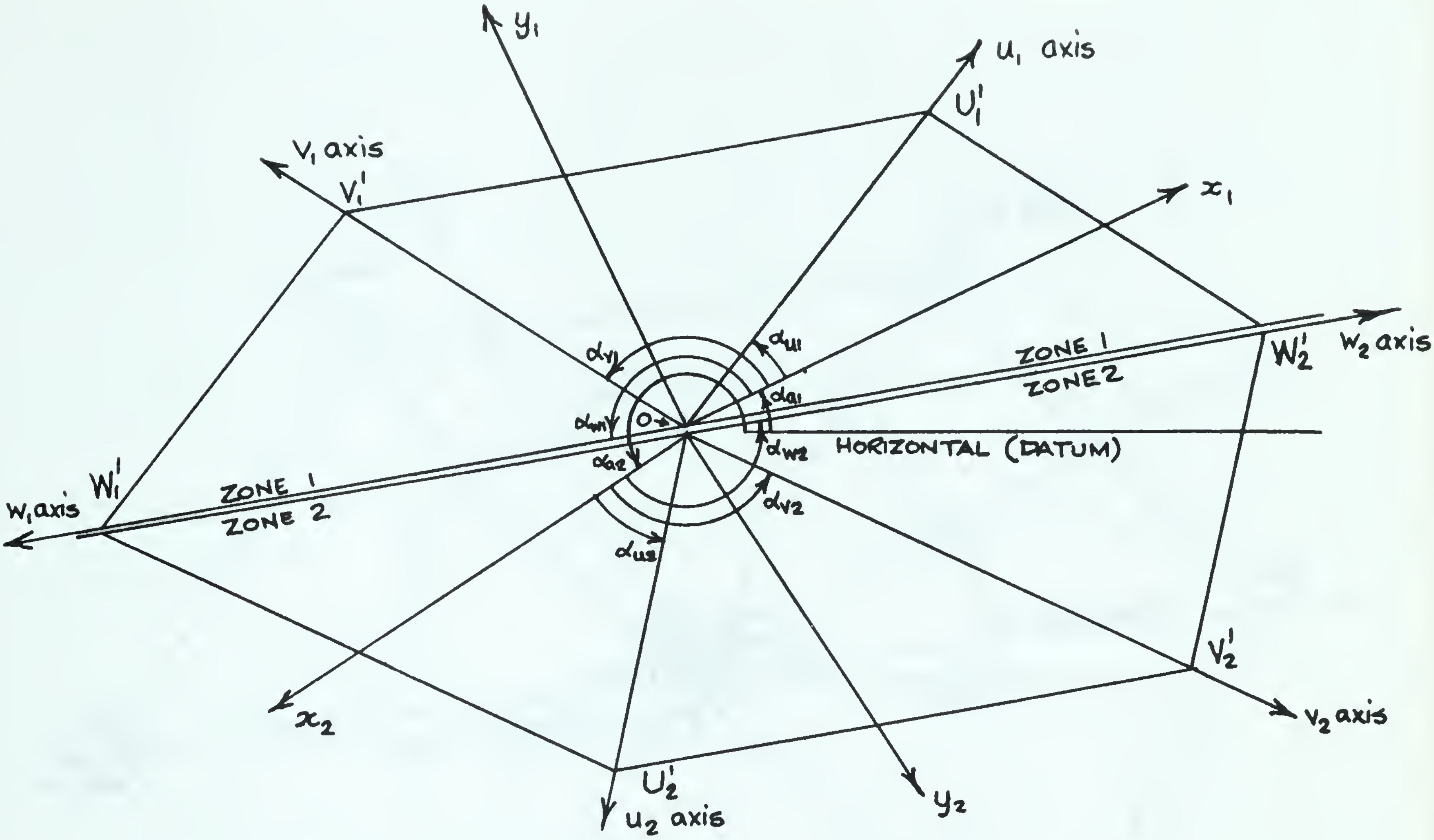

FIGURE B2. Boundary of two zones

Point on a boundary between two zones

Consider a portion of a common boundary between two zones, as shown by the double line in FIGURE B2. The notation used has the same meaning as was used in FIGURE B1 except that a subscript 1 or 2 has been added, according to which of the zones the notation refers. The mesh configuration in each zone is devised so that O, a point on the boundary, is a node common to both meshes. One axis of each mesh lies along the common boundary. In FIGURE B2, the Ow axes of both zones are shown as the boundary axes; in general any axis (Ou, Ov or Ow) of one zone may be matched with any axis of the other zone. Those axes not lying along the boundary (i.e., the Ou and Ov axes of

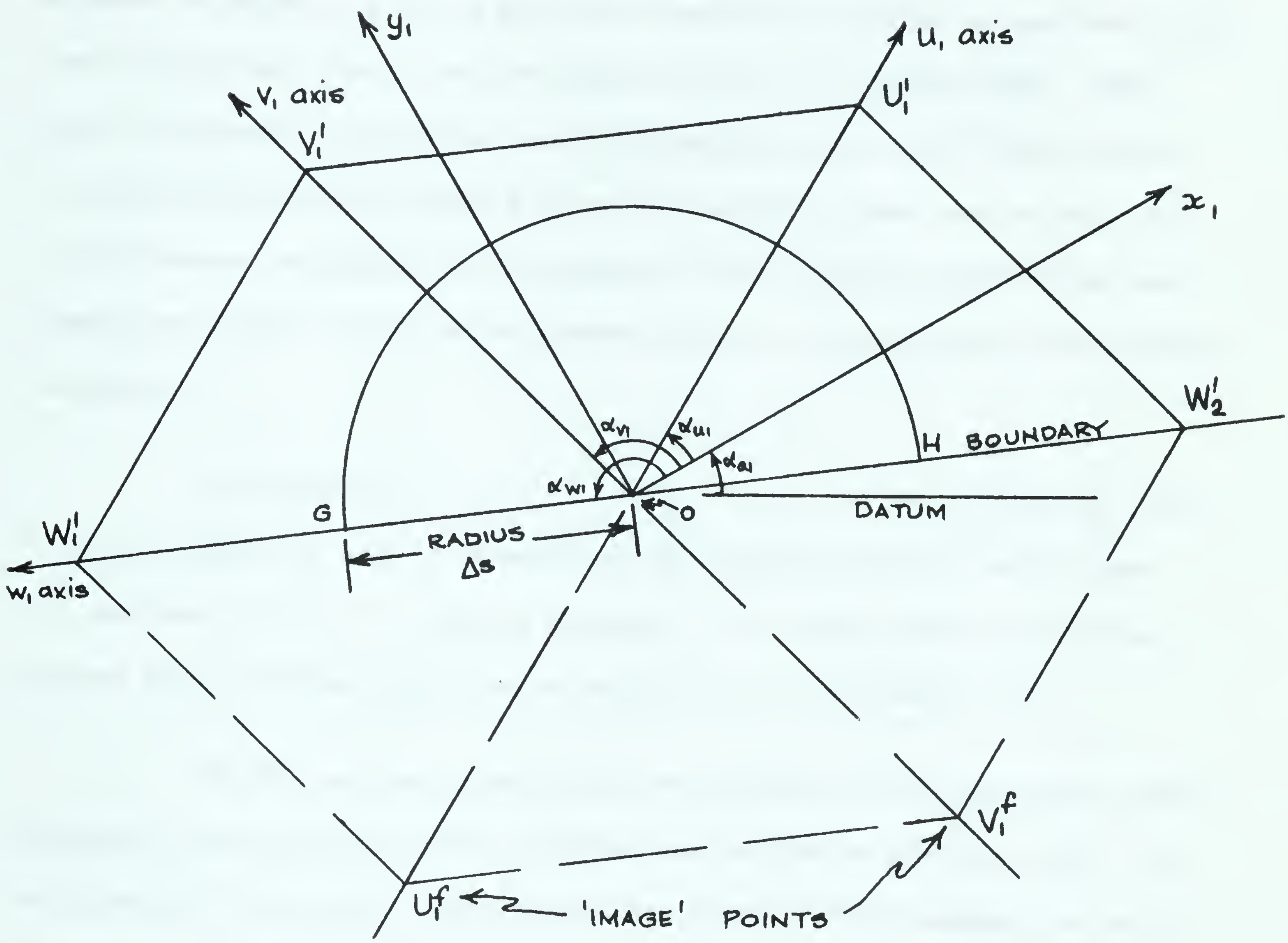

FIGURE B3. Image of zone 1.

both zones in FIGURE B2) do not, in general, lie along the same straight line on either side of the boundary; therefore, although there are six nodes adjacent to point O, a finite difference expression similar to equation (3.16) cannot be stated directly for the axes which are not boundary axes. Each zone is considered separately; two ficticious head values at 'image' points (as shown for zone 1 in FIGURE B3) are introduced for each zone so that finite difference expressions may be stated. These ficticious head values are treated as unknowns which are eliminated later by solving several simultaneous equations.

The boundary is shown in FIGURE B3 with the image points U_1^f and V_1^f drawn opposite U_1' and V_1' respectively by extending the Ou_1 and Ov_1 axes for one mesh length in a negative direction. The image points and the ficticious heads at those points are signified by the superscript 'f'.

By the continuity condition, the outward flow Q_1 across the curved portion of the perimeter of the elemental semi-circle on GH inside zone 1 is equal to the flow Q_2 into zone 1 across the portion of the boundary within GH. The flow Q_1 is expressed by integrating equation (3.10) between $(\alpha_{W1} - \pi)$ and α_{W1}, that is,

$$Q_1 = \int_{\alpha_{W1}-\pi}^{\alpha_{W1}} q \, d\alpha$$

$$= \int_{\alpha_{W1}-\pi}^{\alpha_{W1}} \left\{ V_{x1} \cos\alpha + V_{y1} \sin\alpha \right\} . \Delta s . d\alpha$$

$$- \int_{\alpha_{W1}-\pi}^{\alpha_{W1}} \left\{ \sum (h)_{MM_1} \left[\frac{(k_{x1}+k_{y1}) \sin(\alpha_{N1}+\alpha_{P1}) \cos\alpha \sin\alpha - 2k_{x1} \sin\alpha_{N1} \sin\alpha_{P1} \cos^2\alpha - 2k_{y1} \cos\alpha_{N1} \cos\alpha_{P1} \sin^2\alpha}{2 \sin(\alpha_{N1}-\alpha_{M1}) . \sin(\alpha_{M1}-\alpha_{P1})} \right] \right\} (\Delta s)^2 . d\alpha ,$$

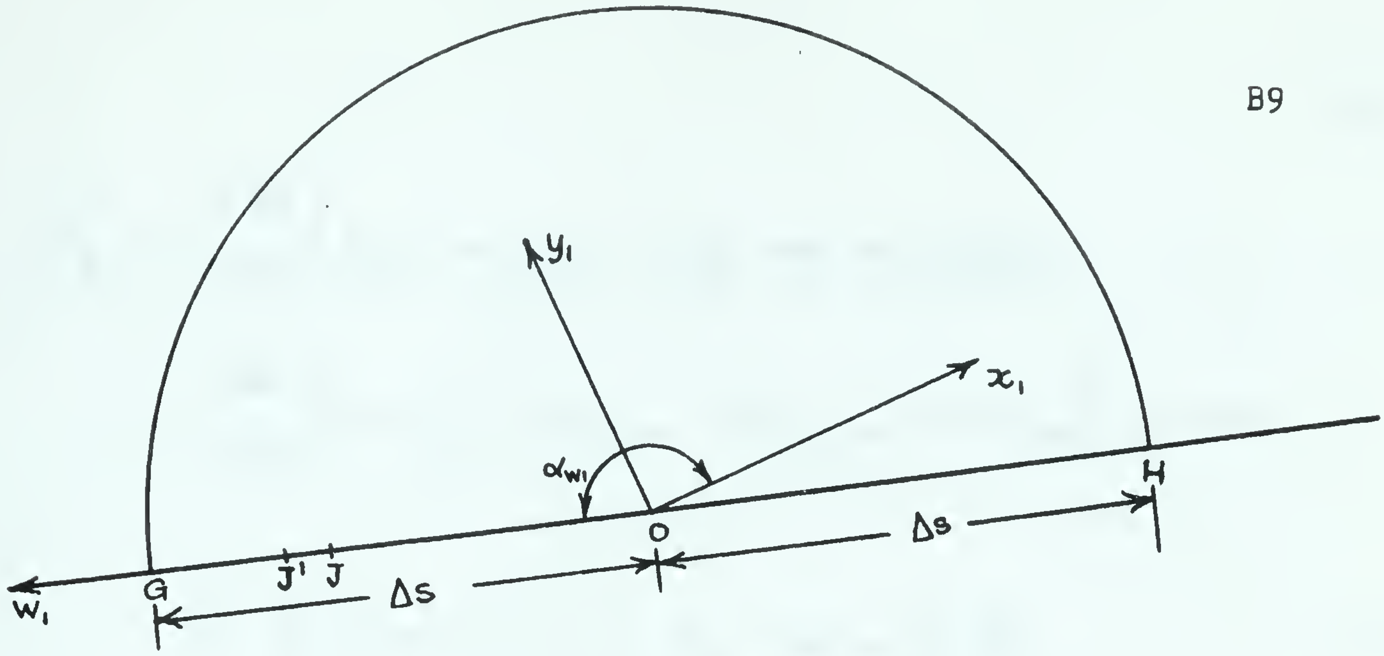

FIGURE B4.

$$
\left.\begin{aligned}
Q_1 &= 2\left\{v_{x1}\sin\alpha_{W1} - v_{y1}\cos\alpha_{W1}\right\}.\Delta s \\
&\quad + \frac{\pi(\Delta s)^2}{2}.\sum (h)_{MM_1}.\frac{k_{x1}\sin\alpha_{N1}\sin\alpha_{P1} + k_{y1}\cos\alpha_{N1}\cos\alpha_{P1}}{\sin(\alpha_{N1}-\alpha_{M1}).\sin(\alpha_{M1}-\alpha_{P1})}.
\end{aligned}\right\}\quad(3.20)
$$

The velocity components at the point J located between G and H on the boundary (FIGURE B4) are equal to the velocity components at O plus the rate of change of the velocity components along Ow_1 multiplied by w (the distance OJ), i.e.,

velocity component at J in x_1 - direction = $v_{x1} + (v_{x1})_w.w$,

velocity component at J in y_1 - direction = $v_{y1} + (v_{y1})_w.w$.

Resolving these components in a direction normal to GH, the flow q_1 across the element JJ' of the length δw is given by

$$
q_1 = \left\{\left(v_{x1} + (v_{x1})_w.w\right)\sin\alpha_{W1} - \left(v_{y1} + (v_{y1})_w.w\right)\cos\alpha_{W1}\right\}.\delta w
$$

$$
\left.\begin{aligned}
&= \left\{v_{x1}\sin\alpha_{W1} - v_{y1}\cos\alpha_{W1}\right\}.\delta w \\
&\qquad + \left\{(v_{x1})_w.\sin\alpha_{W1} - (v_{y1})_w.\cos\alpha_{W1}\right\}.w.\delta w.
\end{aligned}\right\}\quad(3.21)
$$

The flow Q_2 across GH is obtained by integrating equation (3.21) between $-\Delta s$ and $+\Delta s$.

$$Q_2 = \int_{-\Delta s}^{\Delta s} \{ v_{x1} \sin \alpha_{W1} - v_{y1} \cos \alpha_{W1} \}.dw$$

$$+ \int_{-\Delta s}^{\Delta s} \{ (v_{x1})_W . \sin \alpha_{W1} - (v_{y1})_W . \cos \alpha_{W1} \}. W. dw$$

$$= 2 \{ v_{x1} \sin \alpha_{W1} - v_{y1} \cos \alpha_{W1} \}.\Delta s. \qquad (3.22)$$

Since $Q_1 = Q_2$, the right-hand sides of equations (3.20) and (3.22) are equal, and

$$\frac{\pi (\Delta s)^2}{2} \sum (h)_{MM_1} . \frac{k_{x1} \sin \alpha_{N1} . \sin \alpha_{P1} + k_{y1} \cos \alpha_{N1} \cos \alpha_{P1}}{\sin (\alpha_{N1} - \alpha_{M1}). \sin(\alpha_{M1} - \alpha_{P1})} = 0.$$

Therefore,

$$\sum_{M_1 = u_1, v_1, w_1} (h)_{MM_1} . \frac{k_{x1} \sin \alpha_{N1} \sin \alpha_{P1} + k_{y1} \cos \alpha_{N1} \cos \alpha_{P1}}{\sin (\alpha_{N1} - \alpha_{M1}). \sin (\alpha_{M1} - \alpha_{P1})} = 0. \qquad (3.23)$$

Equation (3.23) is the continuity equation for flow inside the semi-circle on GH. This equation can be written in finite difference form, (equation 3.24). The process of reasoning used for obtaining equation (3.24) from equation (3.23) is similar to that for obtaining equation (3.18) from equation (3.11).

$$C_{u_1} . h'_{u_1} + C_{v_1} . h'_{v_1} + C_{W1} . h'_{W1} + C_{u_1} . h^f_{u_1} + C_{v_1} . h^f_{v_1} + C_{W1} . h'_{W2} - 2 (C_{u_1} + C_{v_1} + C_{W1}) h_o = 0, \qquad (3.24)$$

where h'_{u1}, h'_{v1}, h'_{w1}, h'_{w2} are the total heads at U'_1, V'_1, W'_1, W'_2 respectively,

h^f_{u1}, h^f_{v1}, are the fictional heads at U^f_1. V^f_1 respectively

and C_{u1}, C_{v1}, C_{w1} are coefficients appertaining to zone 1 having the same expansions as equations (3.19a, b, c)

A similar equation exists for zone 2; it is:

$$C_{u2}.h'_{u2}+C_{v2}.h'_{v2}+C_{w2}.h'_{w2}+C_{u2}.h^f_{u2}+C_{v2}.h^f_{v2}+C_{w2}.h'_{w1}-2(C_{u2}+C_{v2}+C_{w2})h_o=0. \quad (3.25)$$

In each zone, there are three axes; since two axes are sufficient to locate any point in the plane, any distance along Ov_1 may be regarded as a function of u_1 and w_1 only. Therefore,

$$(h)_{v1} = (h)_{u1}.(u_1)_{v1} + (h)_{w1}.(w_1)_{v1}. \quad (3.26)$$

But, by equations (3.4g) and (3.4h),

$$(u_1)_{v1} = \frac{\sin(\alpha_{w1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})}, \quad (3.27a), \qquad (w_1)_{v1} = -\frac{\sin(\alpha_{u1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})}, \quad (3.27b)$$

By expressing h'_{u1} and h^f_{u1} in the form of a Taylor's series expansion, as was done for h'_u and h''_u in equations (3.12a) and (3.12b), it may be shown that

$$(h)_{u1} = \frac{h'_{u1} - h^f_{u1}}{2\Delta_1 u_1} - \frac{(\Delta_1 u_1)^2}{6}.(h)_{uuu1} - \cdots\cdots$$

Neglecting terms containing $\Delta_1 u_1$ raised to the power of 2 or greater, finite difference expressions for the first derivative of h along Ou_1, Ov_1 and Ow_1 are

$$(h)_{u1} = \frac{h'_{u1} - h^f_{u1}}{2\Delta_1 u_1}, \quad (3.28a)$$

$$(h)_{v1} = \frac{h'_{v1} - h^f_{v1}}{2\Delta_1 v_1}, \quad (3.28b)$$

$$(h)_{w1} = \frac{h'_{w1} - h'_{w2}}{2\Delta_1 w_1}. \quad (3.28c)$$

Substituting equations (3.27a), (3.27b), (3.28a), (3.28b) and (3.28c) in equation (3.26),

$$\frac{h'_{v1}-h^f_{v1}}{2\Delta_1 v_1} = \frac{h'_{u1}-h^f_{u1}}{2\Delta_1 u_1}\cdot\frac{\sin(\alpha_{w1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})} - \frac{h'_{w1}-h'_{w2}}{2\Delta_1 w_1}\cdot\frac{-\sin(\alpha_{u1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})}. \quad (3.29)$$

By the sine rule on triangle $OU'_1V'_1$, (FIGURE B3),

$$\frac{\Delta_1 u_1}{\sin(\alpha_{w1}-\alpha_{v1})} = \frac{\Delta_1 v_1}{\sin(\alpha_{w1}-\alpha_{u1})} = \frac{\Delta_1 w_1}{\sin(\alpha_{v1}-\alpha_{u1})}.$$

Denoting each of the above equalities by F_1, and substituting for $\Delta_1 u_1$, $\Delta_1 v_1$ and $\Delta_1 w_1$ in equation (3.29),

$$\frac{h'_{v1}-h^f_{v1}}{2F_1\sin(\alpha_{w1}-\alpha_{u1})} = \frac{h'_{u1}-h^f_{u1}}{2F_1\sin(\alpha_{w1}-\alpha_{v1})}\cdot\frac{\sin(\alpha_{w1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})} + \frac{h'_{w1}-h'_{w2}}{2F_1\sin(\alpha_{v1}-\alpha_{u1})}\cdot\frac{-\sin(\alpha_{u1}-\alpha_{v1})}{\sin(\alpha_{w1}-\alpha_{u1})}$$

Simplifying,

$$h'_{u1} - h'_{v1} + h'_{w1} - h^f_{u1} + h^f_{v1} - h'_{w2} = 0. \quad (3.30)$$

Similarly, when considering zone 2,

$$h'_{u2} - h'_{v2} + h'_{w2} - h^f_{u2} + h^f_{v2} - h'_{w1} = 0. \quad (3.31)$$

Any distance along Ox_1 and Oy_1 may be regarded as a function of u_1 and w_1; therefore,

$$(h)_{x1} = (h)_{u1}\cdot(u_1)_{x1} + (h)_{w1}\cdot(w_1)_{x1}. \quad (3.32)$$

But, by equations (3.4c) and (3.4d),

$$(u_1)_{x1} = \frac{\sin\alpha_{w1}}{\sin(\alpha_{w1}-\alpha_{u1})}, \quad (3.33a), \qquad (w_1)_{x1} = -\frac{\sin\alpha_{u1}}{\sin(\alpha_{w1}-\alpha_{u1})}, \quad (3.33b)$$

Substituting equations (3.28a), (3.28c), (3.33a) and (3.33b) in equation (3.32),

$$(h)_{x1} = \frac{h'_{u1} - h^f_{u1}}{2\Delta_1 u_1} \cdot \frac{\sin \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} + \frac{h'_{w1} - h'_{w2}}{2\Delta_1 w_1} \cdot \frac{-\sin \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})} .$$

Making the sine rule substitutions,

$$(h)_{x1} = \frac{h'_{u1} - h^f_{u1}}{2F_1 \sin(\alpha_{w1} - \alpha_{v1})} \cdot \frac{\sin \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} + \frac{h'_{w1} - h'_{w2}}{2F_1 \sin(\alpha_{v1} - \alpha_{u1})} \cdot \frac{-\sin \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})}, \quad (3.34)$$

Similarly, using equations (3.4e) and (3.4f),

$$(h)_{y1} = \frac{h'_{u1} - h^f_{u1}}{2F_1 \sin(\alpha_{w1} - \alpha_{v1})} \cdot \frac{-\cos \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} + \frac{h'_{w1} - h'_{w2}}{2F_1 \sin(\alpha_{v1} - \alpha_{u1})} \cdot \frac{\cos \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})}, \quad (3.35)$$

The flow Q_2 across GH was shown (equation 3.22) to be given by

$$Q_2 = 2 \left\{ v_{x1} \sin \alpha_{w1} - v_{y1} \cos \alpha_{w1} \right\} . \Delta s. \quad (3.22)$$

By Darcy's law, $v_{x1} = -k_{x1}(h)_{x1}$ and $v_{y1} = -k_{y1}(h)_{y1}$, therefore

$$Q_2 = -2 \left\{ k_{x1} (h)_{x1} \sin \alpha_{w1} - k_{y1} (h)_{y1} \cos \alpha_{w1} \right\} . \Delta s. \quad (3.36)$$

Substituting equations (3.34) and (3.35) in equation (3.36),

$$Q_2 = -2 \left\{ \frac{h'_{u1} - h^f_{u1}}{2F_1} \cdot \frac{k_{x1} \sin^2 \alpha_{w1} + k_{y1} \cos^2 \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{v1}) \sin(\alpha_{w1} - \alpha_{u1})} - \frac{h_{w1} - h_{w2}}{2F_1 \sin(\alpha_{v1} - \alpha_{u1})} \cdot \frac{k_{x1} \sin \alpha_{w1} \sin \alpha_{u1} + k_{y1} \cos \alpha_{w1} \cos \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})} \right\} . \Delta s, \quad (3.37)$$

By adding the expressions for C_{u1} and C_{v1} (equations 3.19a and 3.19b) it may be shown that

$$C_{u1} + C_{v1} = \frac{k_{x1} \sin^2 \alpha_{w1} + k_{y1} \cos^2 \alpha_{w1}}{\sin(\alpha_{v1} - \alpha_{w1}) . \sin(\alpha_{w1} - \alpha_{u1})} . \sin(\alpha_{v1} - \alpha_{u1}).$$

Equation (3.37) may therefore be more conveniently written as

$$\begin{aligned} Q_2 &= 2 \left\{ \frac{h'_{u1} - h^f_{u1}}{2F_1 \sin(\alpha_{v1} - \alpha_{u1})} . (C_{u1} + C_{v1}) + \frac{h'_{w1} - h'_{w2}}{2F_1 \sin(\alpha_{v1} - \alpha_{u1})} . C_{v1} \right\} . \Delta s \\ &= \left\{ (h'_{u1} - h^f_{u1}) . (C_{u1} + C_{v1}) + (h'_{w1} - h'_{w2}) . C_{v1} \right\} . (\Delta s / \Delta_1 w_1), \quad (3.38) \end{aligned}$$

A further expression for Q_2 may be derived by considering zone 2, i.e.,

$$Q_2 = -\{(h'_{u2} - h^f_{u2}).(C_{u2} + C_{v2}) + (h'_{w2} - h'_{w1}).C_{v2}\}.(\Delta s/\Delta_1 w_2), \quad (3.39)$$

Equating the right-hand sides of equations (3.38) and (3.39), and observing that $\Delta_1 w_1 = \Delta_1 w_2$ (= OW^1_1 = OW^1_2) ,

$$\left.\begin{aligned}(C_{u1} + C_{v1}).(h'_{u1} - h^f_{u1}) + (C_{u2} + C_{v2}).(h'_{u2} - h^f_{u2}) \\ + (C_{v1} - C_{v2}).(h'_{w1} - h'_{w2}) = 0.\end{aligned}\right\} \quad (3.40)$$

The five equations (3.24), (3.25), (3.30), (3.31) and (3.40) are mutually independent, and contain the four ficticious heads h^f_{u1}, h^f_{v1}, h^f_{u2} and h^f_{v2}. These terms may be eliminated by solving the five equations simultaneously. Hence

$$C_{u1}.h'_{u1} + C_{v1}.h'_{v1} + C_{u2}.h'_{u2} + C_{v2}.h'_{v2} + \frac{C_{w1}+C_{w2}}{2}.(h'_{w1} + h'_{w2}) - (C_{u1} + C_{v1} + C_{w1} + C_{u2} + C_{v2} + C_{w2})h_o = 0, \quad (3.41)$$

where C_{u1}, C_{v1}, C_{w1}, C_{u2}, C_{v2}, C_{w2} are coefficients having the expansion written in equations (3.19a), (3.19b) and (3.19c) and referring to the zone signified by the subscript.

Equation (3.41) is the required finite difference equation.

Appendix C

COMPUTER PROGRAM :

PROCEDURES OF SPECIFIC PARTS

APPENDIX C

COMPUTER PROGRAM : PROCEDURES OF SPECIFIC PARTS

Division of an n- sided polygon into triangles

An irregular polygon $P_1 P_2 \ldots P_n$ as shown in FIGURE C1, is specified by the coordinates $(x_1\ y_1)$, $(x_2\ y_2)$,...., $(x_n\ y_n)$ listed in clockwise rotation round the polygon. The directions of each side are calculated previously from the coordinates, paying due regard to quadrant.

A triangle is formed by joining two alternate corner points, (i.e., P_2 to P_4, or P_3 to P_5, etc.). In order that the triangle lie completely within the polygon the three following conditions must apply:

(a) The angle inside the polygon at the intermediate corner point is less than 180 degrees. For example, the triangle formed by joining P_2 to P_4 in FIGURE C1 is inside the polygon since the angle at P_3 is less than 180 degrees; conversely, triangle $P_5 P_6 P_7$ is outside the polygon since the angle inside the polygon at P_6 exceeds 180 degrees.

(b) The angle inside the polygon at the point of the triangle which bears the lowest subscript is greater than the angle inside the triangle at the same point. For example, triangle $P_2 P_3 P_4$ is completely inside the polygon since the polygon angle at P_2 (angle $P_1 P_2 P_3$) is greater than angle $P_4 P_2 P_3$, but triangle $P_7 P_8 P_9$ is not completely inside the polygon since angle $P_6 P_7 P_8$ is not greater than angle $P_9 P_7 P_8$.

(c) The line joining alternate points is not intersected by any of the sides of the polygon which do not pass through those points. Triangle $P_2 P_3 P_4$

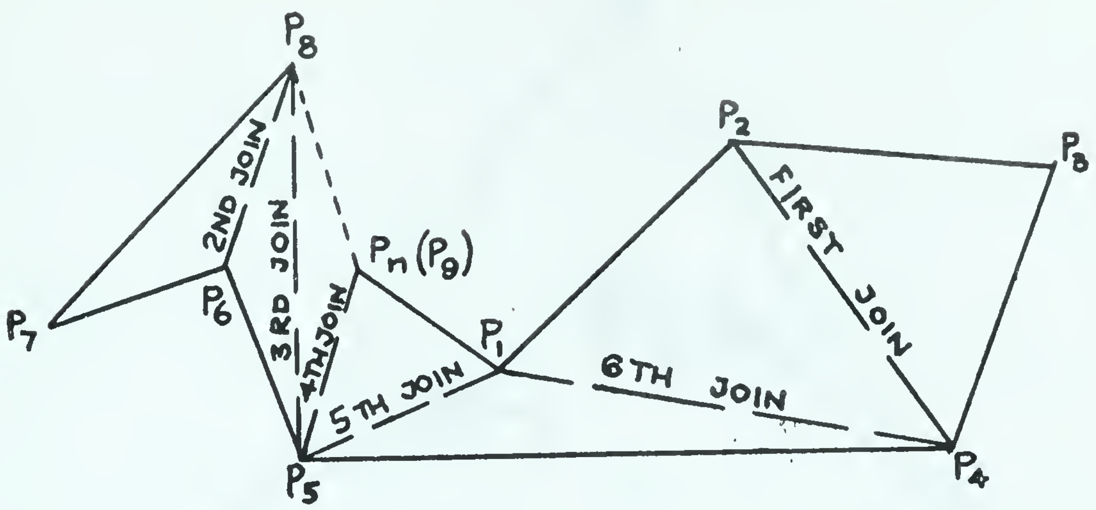

FIGURE C1 Irregular polygon

qualifies on this condition, since the line joining P_2P_4 is not intersected by any of the sides P_5P_6, P_6P_7, P_7P_8, P_8P_9, P_9P_1, but triangle $P_4P_5P_6$ is not completely inside the polygon, since the line P_4P_6 is intersected by P_9P_1 and P_1P_2.

By calculating the direction of the line joining alternate points on the polygon, and using the known directions of the polygon sides, the triangle is easily tested on conditions (a) and (b). The intersection points of the line joining alternate points with each of the other polygon sides, produced if necessary, is calculated; inspection of the positions of these points indicates whether condition (c) is satisfied.

It may appear at first that a triangle which does not satisfy condition (b) could not satisfy condition (c), and therefore that condition (b) is unnecessary. Although this statement is true in most cases, it is not generally true. In FIGURE C2, triangle $P_2P_3P_4$, which is not completely within the polygon, fails to qualify on condition (b), yet satisfies conditions (a) and (c).

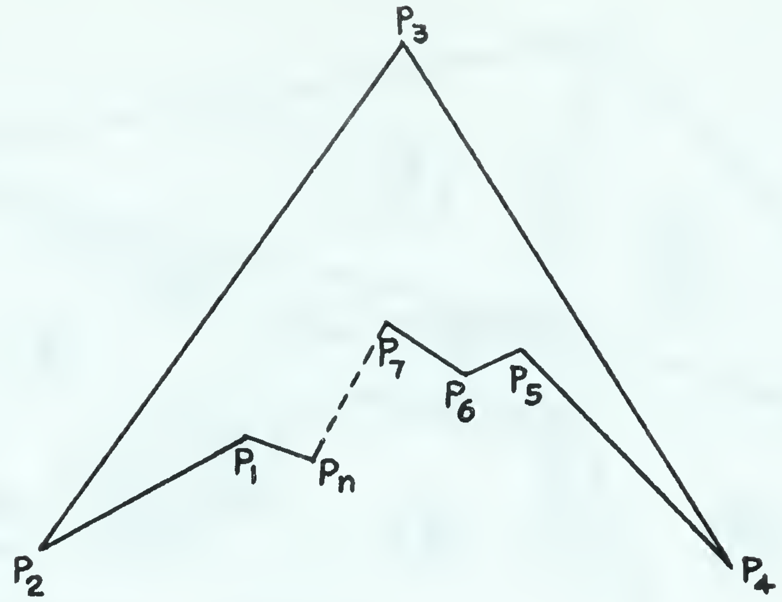

FIGURE C2

When the data on a polygon is ready to be used, an inspection is made to see whether the polygon has only three sides. If so, the coordinates of the corner points are stored as apices of a triangle; the coefficients C_u, C_v, and C_w are calculated, and no further work is required in this part of the program. If the polygon has more than three sides, the following procedure is adopted.

The points P_2 and P_4 are joined and triangle $P_2P_3P_4$ is inspected to see if conditions (a), (b) and (c) are satisfied. If not, P_3 and P_5 are joined and triangle $P_3P_4P_5$ is inspected. If this triangle does not satisfy conditions (a), (b) and (c), triangle $P_4P_5P_6$ is tried, and so on. The geometry of any polygon dictates that, before triangle $P_nP_1P_2$ is reached, a triangle satisfying conditions (a), (b) and (c) is always found. The coefficients C_u, C_v and C_w for this triangle are calculated and stored, together with the coordinates of the triangle apices. Calling this triangle by the points P_r P_{r+1} P_{r+2}, where $2 \leqslant r \leqslant n-1$, the points P_{r+2}, $P_{r+3}, \ldots,$ P_n are re-lettered P_{r+1}, $P_{r+2}, \ldots,$ P_{n-1} respectively; so that the polygon, which is reduced from n to n-1 sides, is lettered

$$P_1P_2 \ldots P_r \text{ and (re-lettered) } P_{r+1}\ P_{r+2} \ldots P_{n-1}.$$

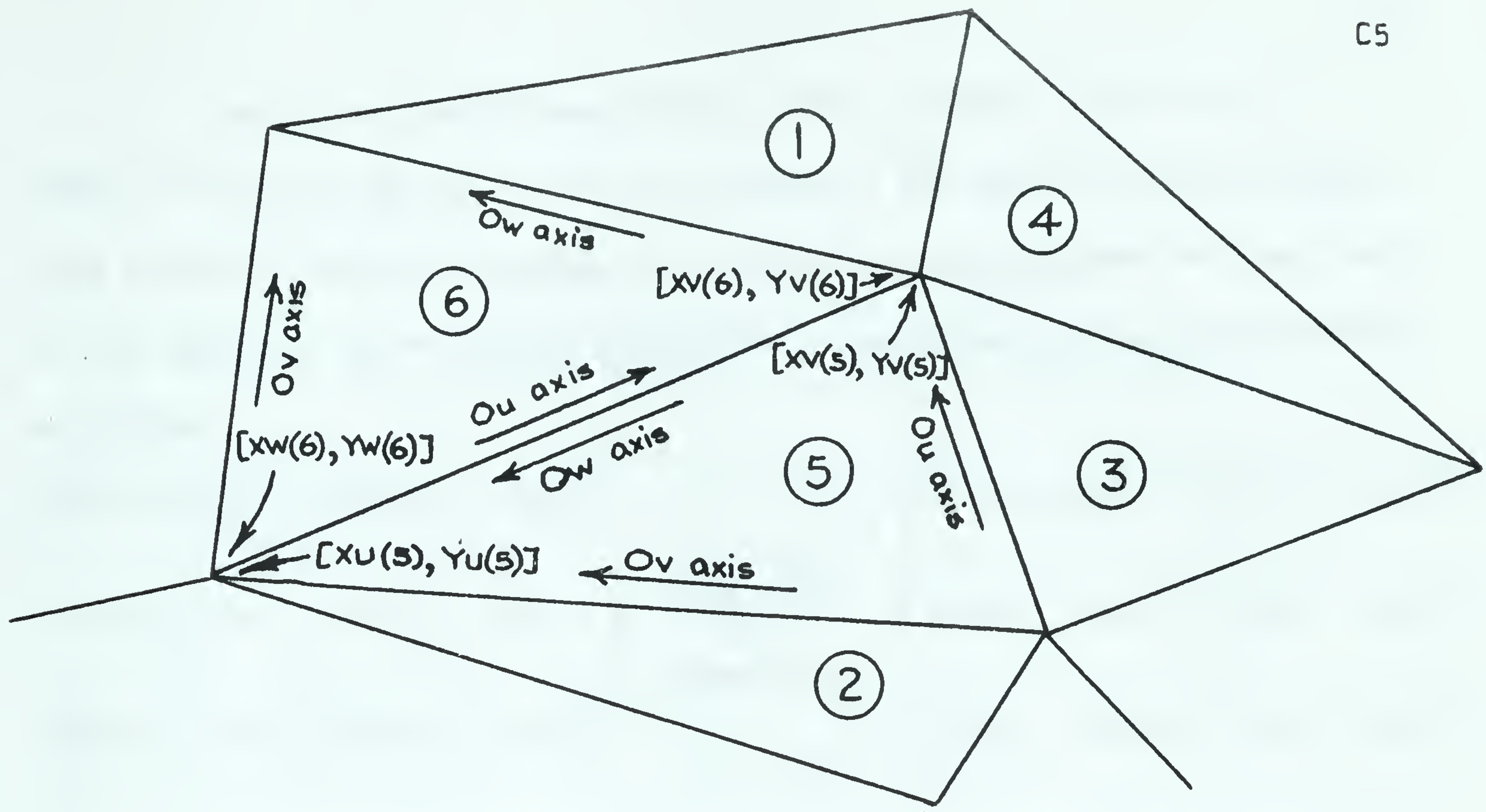

FIGURE C3 Part of a flow region

The procedure is repeated until the number of sides of the polygon is reduced to three and C_u, C_v and C_w are calculated for the remaining triangle.

It may be shown that, when the above procedure is adopted, the polygon in FIGURE C1 is divided into triangles in the succession indicated.

Selection of a triangle side which adjoins a given line

Before a residual R can be calculated for a node on a side of a triangle, the side of the other triangle which lies along the same line must be determined. It is assumed that an inspection has already shown that the triangle side does not lie along a constant potential face or an impermeable boundary of the flow region.

Since the coordinates [XU(I), YU(I)], [XV(I), YV(I)], and [XW(I), YW(I)] of the apices of any triangle I are always listed in clockwise rotation, two of the apices of triangle J are coincident with any two of the apices of an adjoining triangle K in one of only three combinations as follows:

$$\left.\begin{array}{c} [XU(J),\ YU(J)],\ [XV(J),\ YV(J)] \\ \text{or} \\ [XV(J),\ YV(J)],\ [XW(J),\ YW(J)] \\ \text{or} \\ [XW(J),\ YW(J)],\ [XU(J),\ YU(J)] \end{array}\right\} \begin{array}{c} \text{may be} \\ \text{coincident} \\ \text{with,} \\ \text{respectively} \end{array} \left\{\begin{array}{c} [XW(K),\ YW(K)],\ [XV(K),\ YV(K)] \\ \text{or} \\ [XU(K),\ YU(K)],\ [XW(K),\ YW(K)] \\ \text{or} \\ [XV(K),\ YV(K)],\ [XU(K),\ YU(K)] \end{array}\right.$$

If the Ow axis of the given triangle lies along the boundary containing the node under consideration and if the number identifying the triangle is 5, the boundary passes through the points [XU(5), YU(5)] and [XV(5), YV(5)], (FIGURE C3). By observing the possible combinations for matching apices of adjoining triangles, the side of the second triangle which adjoins this boundary is determined as described in flow chart form in FIGURE C4. For the case illustrated in FIGURE C3, the "yes" option is taken at step C3, when K is 6.

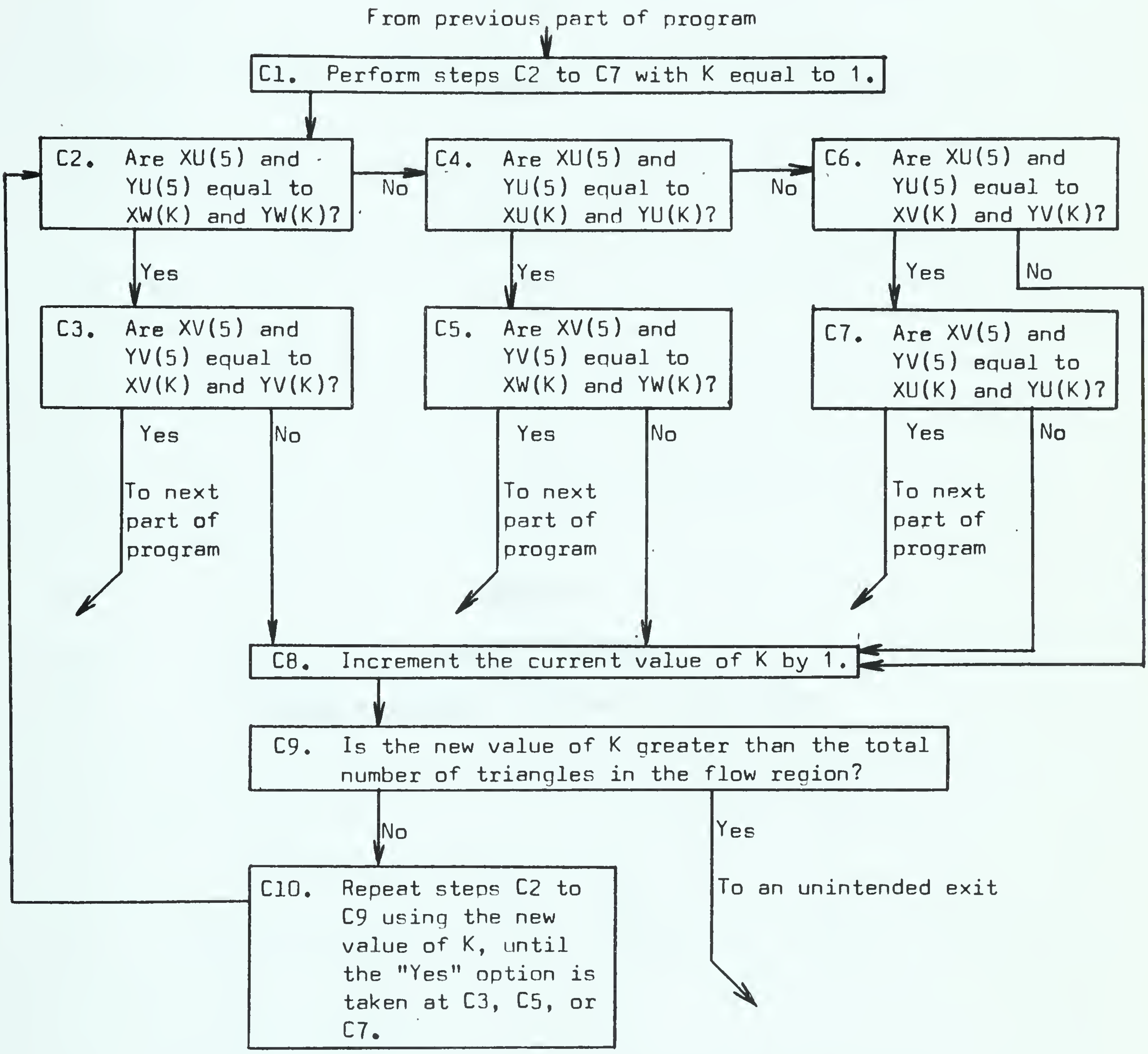

FIGURE C4. Flow chart for selecting a triangle side which adjoins a given line.

Appendix D

COMPUTER PROGRAM :

FORTRAN STATEMENTS AND TYPICAL SET OF DATA

APPENDIX D

COMPUTER PROGRAM : FORTRAN STATEMENTS AND TYPICAL SET OF DATA

FORTRAN statements

The complete list of statements appears on pages D5 to D17. A brief explanation of the more important terms in the program is given below.

A( )	Array containing directions of each side of a polygon in radians.
AA	Direction of minimum permeability in radians.
AADEG	Direction of minimum permeability in degrees.
B	Square root of SM, divided by D.
BM	Current maximum value of B.
CU( ) CV( ) CW( )	Arrays containing coefficients 'C' for use in equations (3.45) to (3.49).
D	Maximum desirable distance between nodes for any zone.
H(, ,)	Potential head at the node signified by the subscripts.
HA	Head acting on a constant potential face.
HAI	Head allocated initially at all nodes except those on a constant potential face.
I	Triangle counter.
IJ	Counter used for identifying triangles when I is in use.
IUI( ) IVI( ) IWI( )	Arrays containing numbers of triangles which lie on impermeable boundaries.
IUU( ) IVV( ) IWW( )	Arrays containing numbers of triangles which lie on constant potential faces.

I1( )	Array containing the lowest triangle number in each zone.
I2	Total number of triangles in the flow region.
I4, I5	Counters used in the format of printed output.
I7	Iterations counter.
J	Zone counter.
JI	Counter used for identifying triangles when I is in use.
J1, J2, J3	Counters.
K, L, M	Counters.
N	Number of nodes on the side of any triangle.
NB	Number of boundaries of any zone.
NIT	Maximum number of iterations allowed.
NZ	Number of zones in the flow region.
P	π
PA	Minimum permeability of any zone.
PB	Maximum permeability of any zone.
R	Residual at any node.
RF	Amount by which the over-relaxation factor exceeds 1.
RM	Current maximum residual in current iteration.
RML	Maximum residual in the previous iteration.
RT	Tolerable residual.
R1, R2, R3	Components of the residual.
S	Square of the length of the side of any triangle.
SM	Current maximum value of S.

TOL TQL	Tolerances.
X( ) Y( )	Arrays containing x- and y- coordinates of corner points of each zone.
XC( ) YC( )	Arrays containing x- and y- coordinates of each node.
XU( ) YU( ) XV( ) YV( ) XW( ) YW( )	Arrays containing x- and y- coordinates of apices of each triangle.
XX YX	Coordinates of intersection point of two lines.
X1 Y1 X2 Y2	Coordinates locating a constant potential face or impermeable boundary.

```
      DIMENSION I1(20),X(21),Y(21),A(20),XU(70),YU(70),XV(70),YV(70)
      DIMENSION XW(70),YW(70),CU(70),CV(70),CW(70),XC(11),YC(11)
      DIMENSION H(70,11,11)
      DIMENSION IUU(30),IVV(30),IWW(30),IUI(30),IVI(30),IWI(30)
      P=3.1415927
      TOL=0.00001
      TQL=0.0001
    2 READ (5,1000) NZ,RT,NIT
      IF(NZ)8,8,4
    4 IF(NZ-20)9,9,6
    6 WRITE (6,1016)
    8 STOP
    9 I=1
      WRITE (6,1001)
      WRITE (6,1002)
      BM=0
      NOK=0
      DO 174 J=1,NZ
      I1(J)=I
      SM=0
      READ (5,1003) D,PA,PB,AADEG
      IF(PB-PA)15,10,15
   10 WRITE (6,1004) J,PA
      GO TO 20
   15 WRITE (6,1005) J,PA,PB,AADEG
   20 AA=AADEG*P/180.
      READ (5,1006) X(1),Y(1)
      NB=2
   25 READ (5,1006) X(NB),Y(NB)
      IF(X(NB)-X(1))35,30,35
   30 IF(Y(NB)-Y(1))35,40,35
   35 NB=NB+1
      IF(NB-21)25,25,36
   36 WRITE (6,1017) J
   37 READ (5,1006) XF,YF
      IF(XF-X(1))37,38,37
   38 IF(YF-Y(1))37,39,37
   39 NOK=1
      GO TO 174
   40 NB=NB-1
      DO 70 K=1,NB
      IF(X(K+1)-X(K))50,45,50
   45 A(K)=P/2.
      IF(Y(K+1)-Y(K))65,65,70
   50 A(K)=ATAN((Y(K+1)-Y(K))/(X(K+1)-X(K)))
      IF(X(K+1)-X(K))65,65,55
   55 IF(Y(K+1)-Y(K))60,70,70
   60 A(K)=A(K)+P
   65 A(K)=A(K)+P
   70 CONTINUE
   75 L=1
      AB=A(3)
      IF(NB-3)148,148,80
   80 L=L+1
      IF(L-NB)84,82,82
```

```
 82 WRITE (6,1018) J
    NOK=1
    GO TO 174
 84 IF(ABS(A(L)-A(L+1))-2.*P+TOL)85,80,80
 85 IF(ABS(A(L)-A(L+1))-TOL)80,80,86
 86 IF(A(L)-A(L+1))89,80,87
 87 IF(ABS(A(L)-A(L+1)-P)-TOL)80,80,88
 88 IF(A(L)-A(L+1)-P)91,80,80
 89 IF(ABS(A(L)-A(L+1)+P)-TOL)80,80,90
 90 IF(A(L)-A(L+1)+P)91,80,80
 91 IF(X(L+2)-X(L))93,92,93
 92 AB=P/2.
    IF(Y(L+2)-Y(L))96,96,97
 93 AB=ATAN((Y(L+2)-Y(L))/(X(L+2)-X(L)))
    IF(X(L+2)-X(L))96,96,94
 94 IF(Y(L+2)-Y(L))95,97,97
 95 AB=AB+P
 96 AB=AB+P
 97 IF(ABS(A(L-1)-A(L))-2.*P+TOL)98,111,111
 98 IF(ABS(A(L-1)-A(L))-TOL)111,111,99
 99 IF(A(L-1)-A(L))102,111,100
100 IF(ABS(A(L-1)-A(L)-P)-TOL)80,80,101
101 IF(A(L-1)-A(L)-P)104,80,111
102 IF(ABS(A(L-1)-A(L)+P)-TOL)80,80,103
103 IF(A(L-1)-A(L)+P)104,80,111
104 IF(ABS(A(L-1)-AB)-2.*P+TOL)105,111,111
105 IF(ABS(A(L-1)-AB)-TOL)111,111,106
106 IF(A(L-1)-AB)109,111,107
107 IF(ABS(A(L-1)-AB-P)-TOL)80,80,108
108 IF(A(L-1)-AB-P)111,80,80
109 IF(ABS(A(L-1)-AB+P)-TOL)80,80,110
110 IF(A(L-1)-AB+P)111,80,80
111 M=L+3
    IF(L-NB+1)113,112,112
112 M=2
    GO TO 115
113 IF(M-NB)116,116,114
114 M=1
115 IF(M-L+2)116,116,148
116 XY=(X(M)-X(M+1))*(Y(L+2)-Y(L))+(X(L+2)-X(L))*(Y(M+1)-Y(M))
    IF(XY)117,146,117
117 DM= X(M)*Y(M+1)-X(M+1)*Y(M)
    DL= X(L+2)*Y(L)-X(L)*Y(L+2)
    YX=(DM*(Y(L+2)-Y(L))+DL*(Y(M+1)-Y(M)))/XY
    XX=(DM*(X(L+2)-X(L))+DL*(X(M+1)-X(M)))/XY
    IF(ABS(Y(M)-Y(M+1))-ABS(X(M)-X(M+1)))123,118,118
118 IF(ABS(YX-Y(M))-TQL)128,128,119
119 IF(ABS(YX-Y(M+1))-TQL)128,128,120
120 IF(YX-Y(M))121,128,122
121 IF(YX-Y(M+1))146,128,128
122 IF(YX-Y(M+1))128,128,146
123 IF(ABS(XX-X(M))-TQL)128,128,124
124 IF(ABS(XX-X(M+1))-TQL)128,128,125
125 IF(XX-X(M))126,128,127
126 IF(XX-X(M+1))146,128,128
```

```
127 IF(XX-X(M+1))128,128,146
128 IF(ABS(Y(L)-Y(L+2))-ABS(X(L)-X(L+2)))134,129,129
129 IF(ABS(YX-Y(L))-TQL)80,80,130
130 IF(ABS(YX-Y(L+2))-TQL)80,80,131
131 IF(YX-Y(L))132,80,133
132 IF(YX-Y(L+2))146,80,80
133 IF(YX-Y(L+2))80,80,146
134 IF(ABS(XX-X(L))-TQL)80,80,135
135 IF(ABS(XX-X(L+2))-TQL)80,80,136
136 IF(XX-X(L))137,80,138
137 IF(XX-X(L+2))146,80,80
138 IF(XX-X(L+2))80,80,146
146 M=M+1
    IF(M-L)115,113,113
148 S=(X(L+2)-X(L))**2 +(Y(L+2)-Y(L))**2
    IF(SM-S)150,152,152
150 SM=S
152 S=(X(L+1)-X(L))**2 +(Y(L+1)-Y(L))**2
    IF(SM-S)154,156,156
154 SM=S
156 S=(X(L+2)-X(L+1))**2 +(Y(L+2)-Y(L+1))**2
    IF(SM-S)158,160,160
158 SM=S
160 XU(I)=X(L+1)
    YU(I)=Y(L+1)
    XV(I)=X(L+2)
    YV(I)=Y(L+2)
    XW(I)=X(L)
    YW(I)=Y(L)
    SN=PA*SIN(A(L)-AA)*SIN(A(L+1)-AA)
    CS=PB*COS(A(L)-AA)*COS(A(L+1)-AA)
    CU(I)=(SN+CS)/SIN(A(L)-A(L+1))
    SN=PA*SIN(A(L+1)-AA)*SIN(AB-AA)
    CS=PB*COS(A(L+1)-AA)*COS(AB-AA)
    CV(I)=(SN+CS)/SIN(A(L+1)-AB)
    SN=PA*SIN(AB-AA)*SIN(A(L)-AA)
    CS=PB*COS(AB-AA)*COS(A(L)-AA)
    CW(I)=(SN+CS)/SIN(AB-A(L))
    I=I+1
    IF(I-70)161,161,167
161 IF(NB-3)170,170,162
162 A(L)=AB
164 IF(L-NB+2)166,166,168
166 X(L+1)=X(L+2)
    Y(L+1)=Y(L+2)
    A(L+1)=A(L+2)
    L=L+1
    GO TO 164
168 X(NB)=X(NB+1)
    Y(NB)=Y(NB+1)
    NB=NB-1
    GO TO 75
167 IF(NB-3)169,169,171
169 IF(J-NZ)171,170,170
171 WRITE (6,1019)
```

```
      STOP
  170 I2=I-1
      B=SQRT(SM)/D
      IF(BM-B)172,174,174
  172 BM=B
  174 CONTINUE
      IF(NOK)8,175,8
  175 N=BM+2.
      IF(N-11)178,178,176
  176 N=11
  178 DO 184 I=1,I2
      DO 182 K1=1,N
      NI=N-K1+1
      DO 180 K2=1,NI
  180 H(I,K1,K2)=-999.
  182 CONTINUE
  184 CONTINUE
      WRITE (6,1007)
      WRITE (6,1008)
      I3=0
      J1=1
      J2=1
      J3=1
      I4=0
      HAA=0
  186 READ (5,1009) X1,Y1,X2,Y2,HA
      IF(ABS(HA+999.)-TQL)189,189,187
  187 I4=I4+1
      HAA=HAA+HA
      IF(X2-X1)196,188,196
  188 IF(Y2-Y1)196,190,196
  189 WRITE (6,1020)
      STOP
  190 J11=J1-1
      J21=J2-1
      J31=J3-1
      TOT=I4-1
      HAI=HAA/TOT
      DO 195 I=1,I2
      DO 193 K1=1,N
      NI=N-K1+1
      DO 191 K2=1,NI
      IF(ABS(H(I,K1,K2)+999.)-TQL)185,185,191
  185 H(I,K1,K2)=HAI
  191 CONTINUE
  193 CONTINUE
  195 CONTINUE
      WRITE (6,1011)
      WRITE (6,1008)
      I3=1
      J1=1
      J2=1
      J3=1
  192 READ (5,1012) X1,Y1,X2,Y2
      IF(X2-X1)196,194,196
```

```
194 IF(Y2-Y1)196,246,196
196 WRITE (6,1010) X1,Y1,X2,Y2
197 DO 198 I=1,I2
    IF(X1-XU(I))206,200,206
200 IF(Y1-YU(I))206,202,206
202 IF(X2-XV(I))198,204,198
204 IF(Y2-YV(I))198,222,198
206 IF(X1-XV(I))214,208,214
208 IF(Y1-YV(I))214,210,214
210 IF(X2-XW(I))198,212,198
212 IF(Y2-YW(I))198,230,198
214 IF(X1-XW(I))198,216,198
216 IF(Y1-YW(I))198,218,198
218 IF(X2-XU(I))198,220,198
220 IF(Y2-YU(I))198,238,198
198 CONTINUE
199 WRITE (6,1020)
    STOP
222 IF(J1-30)223,223,171
223 IF(I3)228,224,228
224 DO 226 K1=1,N
    NI=N-K1+1
226 H(I,K1,NI)=HA
    IWW(J1)=I
    J1=J1+1
    GO TO 243
228 IWI(J1)=I
    J1=J1+1
    GO TO 192
230 IF(J2-30)231,231,171
231 IF(I3)236,232,236
232 DO 234 K1=1,N
234 H(I,K1,1)=HA
    IUU(J2)=I
    J2=J2+1
    GO TO 243
236 IUI(J2)=I
    J2=J2+1
    GO TO 192
238 IF(J3-30)239,239,171
239 IF(I3)244,240,244
240 DO 242 K2=1,N
242 H(I,1,K2)=HA
    IVV(J3)=I
    J3=J3+1
243 DO 233 I=1,I2
    IF(X1-XU(I))203,201,203
201 IF(Y1-YU(I))203,207,203
203 IF(X2-XU(I))209,205,209
205 IF(Y2-YU(I))209,207,209
207 H(I,1,N)=HA
    GO TO 233
209 IF(X1-XV(I))213,211,213
211 IF(Y1-YV(I))213,217,213
213 IF(X2-XV(I))219,215,219
```

```
215 IF(Y2-YV(I))219,217,219
217 H(I,N,1)=HA
    GO TO 233
219 IF(X1-XW(I))225,221,225
221 IF(Y1-YW(I))225,229,225
225 IF(X2-XW(I))233,227,233
227 IF(Y2-YW(I))233,229,233
229 H(I,1,1)=HA
233 CONTINUE
    GO TO 186
244 IVI(J3)=I
    J3=J3+1
    GO TO 192
246 J12=J1-1
    J22=J2-1
    J32=J3-1
    I7=0
    RF=0.8
247 RM=0
    DO 586 I=1,I2
    DO 584 K1=1,N
    NI=N-K1+1
    DO 582 K2=1,NI
    I3=0
248 IF(K1+K2-N-1)250,326,250
250 IF(K1-1)252,290,252
252 IF(K2-1)254,256,254
254 R1=CU(I)*(H(I,K1+1,K2)+H(I,K1-1,K2))
    R2=CV(I)*(H(I,K1,K2+1)+H(I,K1,K2-1))
    R3=CW(I)*(H(I,K1+1,K2-1)+H(I,K1-1,K2+1))
    R          =(R1+R2+R3)/(2.*(CU(I)+CV(I)+CW(I)))-H(I,K1,K2)
    GO TO 578
256 J2=0
258 J2=J2+1
    IF(J21-J2)262,260,260
260 IF(IUU(J2)-I)258,574,258
262 J2=0
264 J2=J2+1
    IF(J22-J2)270,266,266
266 IF(IUI(J2)-I)264,268,264
268 R1=CU(I)*(H(I,K1-1,1)+H(I,K1+1,1))/2.
    R2=CV(I)*H(I,K1,2)+CW(I)*H(I,K1-1,2)
    GO TO 366
270 NK1=N-K1
    DO 272 IJ=1,I2
    IF(XV(I)-XU(IJ))276,273,276
273 IF(YV(I)-YU(IJ))276,274,276
274 IF(XW(I)-XW(IJ))272,275,272
275 IF(YW(I)-YW(IJ))272,284,272
276 IF(XV(I)-XV(IJ))280,277,280
277 IF(YV(I)-YV(IJ))280,278,280
278 IF(XW(I)-XU(IJ))272,279,272
279 IF(YW(I)-YU(IJ))272,286,272
280 IF(XV(I)-XW(IJ))272,281,272
281 IF(YV(I)-YW(IJ))272,282,272
```

```
282 IF(XW(I)-XV(IJ))272,283,272
283 IF(YW(I)-YV(IJ))272,288,272
272 CONTINUE
    GO TO 199
284 R1=(CU(I)+CV(IJ))*(H(I,K1-1,1)+H(IJ,1,K1+1))/2.
    R2=CV(I)*H(I,K1,2)+CW(IJ)*H(IJ,2,K1-1)
    R3=CW(I)*H(I,K1-1,2)+CU(IJ)*H(IJ,2,K1)
    GO TO 364
286 R1=(CU(I)+CW(IJ))*(H(I,K1-1,1)+H(IJ,K1+1,NK1))/2.
    R2=CV(I)*H(I,K1,2)+CU(IJ)*H(IJ,K1-1,NK1+1)
    R3=CW(I)*H(I,K1-1,2)+CV(IJ)*H(IJ,K1,NK1)
    GO TO 364
288 R1=(CU(I)+CU(IJ))*(H(I,K1-1,1)+H(IJ,NK1,1))/2.
    R2=CV(I)*H(I,K1,2)+CV(IJ)*H(IJ,NK1+1,2)
    R3=CW(I)*H(I,K1-1,2)+CW(IJ)*H(IJ,NK1,2)
    GO TO 364
290 IF(K2-1)292,368,292
292 J3=0
294 J3=J3+1
    IF(J31-J3)298,296,296
296 IF(IVV(J3)-I)294,574,294
298 J3=0
300 J3=J3+1
    IF(J32-J3)306,302,302
302 IF(IVI(J3)-I)300,304,300
304 R1=CV(I)*(H(I,1,K2+1)+H(I,1,K2-1))/2.
    R2=CU(I)*H(I,2,K2)+CW(I)*H(I,2,K2-1)
    GO TO 366
306 NK2=N-K2
    DO 308 IJ=1,I2
    IF(XW(I)-XV(IJ))312,309,312
309 IF(YW(I)-YV(IJ))312,310,312
310 IF(XU(I)-XU(IJ))308,311,308
311 IF(YU(I)-YU(IJ))308,320,308
312 IF(XW(I)-XW(IJ))316,313,316
313 IF(YW(I)-YW(IJ))316,314,316
314 IF(XU(I)-XV(IJ))308,315,308
315 IF(YU(I)-YV(IJ))308,322,308
316 IF(XW(I)-XU(IJ))308,317,308
317 IF(YW(I)-YU(IJ))308,318,308
318 IF(XU(I)-XW(IJ))308,319,308
319 IF(YU(I)-YW(IJ))308,324,308
308 CONTINUE
    GO TO 199
320 R1=(CV(I)+CW(IJ))*(H(I,1,K2+1)+H(IJ,NK2+2,K2-1))/2.
    R2=CW(I)*H(I,2,K2-1)+CU(IJ)*H(IJ,NK2,K2)
    R3=CU(I)*H(I,2,K2)+CV(IJ)*H(IJ,NK2+1,K2-1)
    GO TO 364
322 R1=(CV(I)+CU(IJ))*(H(I,1,K2+1)+H(IJ,K2-1,1))/2.
    R2=CW(I)*H(I,2,K2-1)+CV(IJ)*H(IJ,K2,2)
    R3=CU(I)*H(I,2,K2)+CW(IJ)*H(IJ,K2-1,2)
    GO TO 364
324 R1=(CV(I)+CV(IJ))*(H(I,1,K2+1)+H(IJ,1,NK2+2))/2.
    R2=CW(I)*H(I,2,K2-1)+CW(IJ)*H(IJ,2,NK2)
    R3=CU(I)*H(I,2,K2)+CU(IJ)*H(IJ,2,NK2+1)
```

```
      GO TO 364
  326 IF(K1-1)328,436,328
  328 IF(K2-1)330,504,330
  330 J1=0
  332 J1=J1+1
      IF(J11-J1)336,334,334
  334 IF(IWW(J1)-I)332,574,332
  336 J1=0
  338 J1=J1+1
      IF(J12-J1)344,340,340
  340 IF(IWI(J1)-I)338,342,338
  342 R1=CW(I)*(H(I,K1-1,K2+1)+H(I,K1+1,K2-1))/2.
      R2=CU(I)*H(I,K1-1,K2)+CV(I)*H(I,K1,K2-1)
      GO TO 366
  344 DO 346 IJ=1,I2
      IF(XU(I)-XW(IJ))350,347,350
  347 IF(YU(I)-YW(IJ))350,348,350
  348 IF(XV(I)-XV(IJ))346,349,346
  349 IF(YV(I)-YV(IJ))346,358,346
  350 IF(XU(I)-XU(IJ))354,351,354
  351 IF(YU(I)-YU(IJ))354,352,354
  352 IF(XV(I)-XW(IJ))346,353,346
  353 IF(YV(I)-YW(IJ))346,360,346
  354 IF(XU(I)-XV(IJ))346,355,346
  355 IF(YU(I)-YV(IJ))346,356,346
  356 IF(XV(I)-XU(IJ))346,357,346
  357 IF(YV(I)-YU(IJ))346,362,346
  346 CONTINUE
      GO TO 199
  358 R1=(CW(I)+CU(IJ))*(H(I,K1+1,K2-1)+H(IJ,K1-1,1))/2.
      R2=CU(I)*H(I,K1-1,K2)+CV(IJ)*H(IJ,K1,2)
      R3=CV(I)*H(I,K1,K2-1)+CW(IJ)*H(IJ,K1-1,2)
      GO TO 364
  360 R1=(CW(I)+CV(IJ))*(H(I,K1+1,K2-1)+H(IJ,1,K2+1))/2.
      R2=CU(I)*H(I,K1-1,K2)+CW(IJ)*H(IJ,2,K2-1)
      R3=CV(I)*H(I,K1,K2-1)+CU(IJ)*H(IJ,2,K2)
      GO TO 364
  362 R1=(CW(I)+CW(IJ))*(H(I,K1+1,K2-1)+H(IJ,K2+1,K1-1))/2.
      R2=CU(I)*H(I,K1-1,K2)+CU(IJ)*H(IJ,K2-1,K1)
      R3=CV(I)*H(I,K1,K2-1)+CV(IJ)*H(IJ,K2,K1-1)
  364 R          =(R1+R2+R3)/(CU(I)+CV(I)+CW(I)+CU(IJ)+CV(IJ)+CW(IJ))
      R          =R          -H(I,K1,K2)
      GO TO 578
  366 R          =(R1+R2)/(CU(I)+CV(I)+CW(I))-H(I,K1,K2)
      GO TO 578
  368 IF(I3)402,370,402
  370 R1=0
      R2=0
      IJ=I
  372 JI=IJ
      J2=0
  374 J2=J2+1
      IF(J21-J2)378,376,376
  376 IF(IUU(J2)-JI)374,574,374
  378 R1=R1+CU(JI)*H(JI,2,1)+CV(JI)*H(JI,1,2)
```

```
    R2=R2+CU(JI)+CV(JI)
    J2=0
380 J2=J2+1
    IF(J22-J2)384,382,382
382 IF(IUI(J2)-JI)380,572,380
384 DO 386 IJ=1,I2
    IF(XV(JI)-XU(IJ))391,387,391
387 IF(YV(JI)-YU(IJ))391,388,391
388 IF(XW(JI)-XW(IJ))386,389,386
389 IF(YW(JI)-YW(IJ))386,390,386
390 IF(IJ-I)372,576,372
391 IF(XV(JI)-XV(IJ))396,392,396
392 IF(YV(JI)-YV(IJ))396,393,396
393 IF(XW(JI)-XU(IJ))386,394,386
394 IF(YW(JI)-YU(IJ))386,395,386
395 IF(IJ-I)440,576,440
396 IF(XV(JI)-XW(IJ))386,397,386
397 IF(YV(JI)-YW(IJ))386,398,386
398 IF(XW(JI)-XV(IJ))386,399,386
399 IF(YW(JI)-YV(IJ))386,400,386
400 IF(IJ-I)508,576,508
386 CONTINUE
    GO TO 199
402 IJ=I
404 JI=IJ
    J3=0
406 J3=J3+1
    IF(J31-J3)410,408,408
408 IF(IVV(J3)-JI)406,574,406
410 IF(I3)414,412,414
412 R1=R1+CV(JI)*H(JI,1,2)+CU(JI)*H(JI,2,1)
    R2=R2+CV(JI)+CU(JI)
    GO TO 416
414 I3=0
416 J3=0
418 J3=J3+1
    IF(J32-J3)422,420,420
420 IF(IVI(J3)-JI)418,576,418
422 DO 424 IJ=1,I2
    IF(XW(JI)-XV(IJ))428,425,428
425 IF(YW(JI)-YV(IJ))428,426,428
426 IF(XU(JI)-XU(IJ))424,427,424
427 IF(YU(JI)-YU(IJ))424,540,424
428 IF(XW(JI)-XW(IJ))432,429,432
429 IF(YW(JI)-YW(IJ))432,430,432
430 IF(XU(JI)-XV(IJ))424,431,424
431 IF(YU(JI)-YV(IJ))424,404,424
432 IF(XW(JI)-XU(IJ))424,433,424
433 IF(YW(JI)-YU(IJ))424,434,424
434 IF(XU(JI)-XW(IJ))424,435,424
435 IF(YU(JI)-YW(IJ))424,472,424
424 CONTINUE
    GO TO 199
436 IF(I3)470,438,470
438 R1=0
```

```
    R2=0
    IJ=I
440 JI=IJ
    J3=0
442 J3=J3+1
    IF(J31-J3)446,444,444
444 IF(IVV(J3)-JI)442,574,442
446 R1=R1+CV(JI)*H(JI,1,N-1)+CW(JI)*H(JI,2,N-1)
    R2=R2+CV(JI)+CW(JI)
    J3=0
448 J3=J3+1
    IF(J32-J3)452,450,450
450 IF(IVI(J3)-JI)448,572,448
452 DO 454 IJ=1,I2
    IF(XW(JI)-XV(IJ))459,455,459
455 IF(YW(JI)-YV(IJ))459,456,459
456 IF(XU(JI)-XU(IJ))454,457,454
457 IF(YU(JI)-YU(IJ))454,458,454
458 IF(IJ-I)440,576,440
459 IF(XW(JI)-XW(IJ))464,460,464
460 IF(YW(JI)-YW(IJ))464,461,464
461 IF(XU(JI)-XV(IJ))454,462,454
462 IF(YU(JI)-YV(IJ))454,463,454
463 IF(IJ-I)508,576,508
464 IF(XW(JI)-XU(IJ))454,465,454
465 IF(YW(JI)-YU(IJ))454,466,454
466 IF(XU(JI)-XW(IJ))454,467,454
467 IF(YU(JI)-YW(IJ))454,468,454
468 IF(IJ-I)372,576,372
454 CONTINUE
    GO TO 199
470 IJ=I
472 JI=IJ
    J1=0
474 J1=J1+1
    IF(J11-J1)478,476,476
476 IF(IWW(J1)-JI)474,574,474
478 IF(I3)482,480,482
480 R1=R1+CW(JI)*H(JI,2,N-1)+CV(JI)*H(JI,1,N-1)
    R2=R2+CW(JI)+CV(JI)
    GO TO 484
482 I3=0
484 J1=0
486 J1=J1+1
    IF(J12-J1)490,488,488
488 IF(IWI(J1)-JI)486,576,486
490 DO 492 IJ=1,I2
    IF(XU(JI)-XW(IJ))496,493,496
493 IF(YU(JI)-YW(IJ))496,494,496
494 IF(XV(JI)-XV(IJ))492,495,492
495 IF(YV(JI)-YV(IJ))492,404,492
496 IF(XU(JI)-XU(IJ))500,497,500
497 IF(YU(JI)-YU(IJ))500,498,500
498 IF(XV(JI)-XW(IJ))492,499,492
499 IF(YV(JI)-YW(IJ))492,472,492
```

```
500 IF(XU(JI)-XV(IJ))492,501,492
501 IF(YU(JI)-YV(IJ))492,502,492
502 IF(XV(JI)-XU(IJ))492,503,492
503 IF(YV(JI)-YU(IJ))492,540,492
492 CONTINUE
    GO TO 199
504 IF(I3)538,506,538
506 R1=0
    R2=0
    IJ=I
508 JI=IJ
    J1=0
510 J1=J1+1
    IF(J11-J1)514,512,512
512 IF(IWW(J1)-JI)510,574,510
514 R1=R1+CW(JI)*H(JI,N-1,2)+CU(JI)*H(JI,N-1,1)
    R2=R2+CW(JI)+CU(JI)
    J1=0
516 J1=J1+1
    IF(J12-J1)520,518,518
518 IF(IWI(J1)-JI)516,572,516
520 DO 522 IJ=1,I2
    IF(XU(JI)-XW(IJ))527,523,527
523 IF(YU(JI)-YW(IJ))527,524,527
524 IF(XV(JI)-XV(IJ))522,525,522
525 IF(YV(JI)-YV(IJ))522,526,522
526 IF(IJ-I)508,576,508
527 IF(XU(JI)-XU(IJ))532,528,532
528 IF(YU(JI)-YU(IJ))532,529,532
529 IF(XV(JI)-XW(IJ))522,530,522
530 IF(YV(JI)-YW(IJ))522,531,522
531 IF(IJ-I)372,576,372
532 IF(XU(JI)-XV(IJ))522,533,522
533 IF(YU(JI)-YV(IJ))522,534,522
534 IF(XV(JI)-XU(IJ))522,535,522
535 IF(YV(JI)-YU(IJ))522,536,522
536 IF(IJ-I)440,576,440
522 CONTINUE
    GO TO 199
538 IJ=I
540 JI=IJ
    J2=0
542 J2=J2+1
    IF(J21-J2)546,544,544
544 IF(IUU(J2)-JI)542,574,542
546 IF(I3)550,548,550
548 R1=R1+CU(JI)*H(JI,N-1,1)+CW(JI)*H(JI,N-1,2)
    R2=R2+CU(JI)+CW(JI)
    GO TO 552
550 I3=0
552 J2=0
554 J2=J2+1
    IF(J22-J2)558,556,556
556 IF(IUI(J2)-JI)554,576,554
558 DO 560 IJ=1,I2
```

```
      IF(XV(JI)-XU(IJ))564,561,564
  561 IF(YV(JI)-YU(IJ))564,562,564
  562 IF(XW(JI)-XW(IJ))560,563,560
  563 IF(YW(JI)-YW(IJ))560,472,560
  564 IF(XV(JI)-XV(IJ))568,565,568
  565 IF(YV(JI)-YV(IJ))568,566,568
  566 IF(XW(JI)-XU(IJ))560,567,560
  567 IF(YW(JI)-YU(IJ))560,540,560
  568 IF(XV(JI)-XW(IJ))560,569,560
  569 IF(YV(JI)-YW(IJ))560,570,560
  570 IF(XW(JI)-XV(IJ))560,571,560
  571 IF(YW(JI)-YV(IJ))560,404,560
  560 CONTINUE
      GO TO 199
  572 I3=1
      GO TO 248
  574 R=0
      GO TO 578
  576 R             =R1/R2-H(I,K1,K2)
  578 IF(RM-ABS(R))580,582,582
  580 RM=ABS(R)
  582 H(I,K1,K2)=H(I,K1,K2)+R*(1.0+RF)
  584 CONTINUE
  586 CONTINUE
      IF(I7-1)592,592,588
  588 IF(RM-RML)592,592,590
  590 RF=RF*0.8
  592 RML=RM
      I7=I7+1
      IF(I7-NIT)593,594,594
  593 IF(RT-RM)247,594,594
  594 WRITE (6,1022) I7,RM
      J=1
      I4=2
      I5=50
      DO 604 I=1,I2
      IF(J-NZ)595,595,598
  595 IF(I1(J)-1)598,596,598
  596 IF(I5-49)599,599,597
  597 WRITE (6,1013) J,I4
      I4=I4+1
      WRITE (6,1014)
      I5=5
      GO TO 601
  599 WRITE (6,1021)
      WRITE (6,1024) J
      WRITE (6,1014)
      I5=I5+5
  601 JC=J
      J=J+1
  598 DO 602 K1=1,N
      NI=N-K1+1
      DO 600 K2=1,NI
      EK1=K1-1
      EK2=K2-1
```

```
      EN=N-1
      XC(K2)=XW(I)+(XV(I)-XW(I))*EK1/EN+(XU(I)-XW(I))*EK2/EN
  600 YC(K2)=YW(I)+(YV(I)-YW(I))*EK1/EN+(YU(I)-YW(I))*EK2/EN
      WRITE (6,1015) (XC(K2),YC(K2),H(I,K1,K2),K2=1,NI)
      IF(NI-8)606,606,608
  606 IF(NI-4)612,612,610
  608 I5=I5+1
  610 I5=I5+1
  612 I5=I5+1
      IF(I5-56)602,602,614
  614 IF(N-K1)616,616,620
  616 IF(I2-I)604,604,617
  617 IF(J-NZ)618,618,620
  618 IF(I1(J)-I-1)620,604,620
  620 WRITE (6,1023) JC,I4
      I4=I4+1
      WRITE (6,1014)
      I5=5
  602 CONTINUE
      IF(I5-5)622,604,622
  622 WRITE (6,1021)
      I5=I5+1
  604 CONTINUE
      GO TO 2
 1000 FORMAT(7XI3,F10.3,I10)
 10010FORMAT(1H141X4HZONE8X7HMINIMUM8X7HMAXIMUM5X9HDIRECTION4X
     18HPAGE    1)
 1002 FORMAT(1H 43X46HNO    PERMEABILITY   PERMEABILITY   OF MIN PERM/)
 1003 FORMAT(1XF9.2,2E10.2,F10.2)
 1004 FORMAT(1H 42XI3,1PE15.3,3X9HISOTROPIC)
 1005 FORMAT(1H 42XI3,1P2E15.3,0PF14.2)
 1006 FORMAT(1XF9.2,F10.2)
 1007 FORMAT(1H545X40HLOCATIONS OF FACES OF CONSTANT POTENTIAL/)
 1008 FORMAT(1H 47X1HX9X,1HY14X,1HX9X,1HY/)
 1009 FORMAT(1XF9.2,4F10.2)
 1010 FORMAT(1H 40X2F10.2,5X2F10.2)
 1011 FORMAT(1H748X35HLOCATIONS OF IMPERMEABLE BOUNDARIES/)
 1012 FORMAT(1XF9.2,3F10.2)
 1013 FORMAT(1H162X4HZONEI3,42X4HPAGEI4/)
 10140FORMAT(1H 9X1HX9X1HY7X4HHEAD11X1HX9X1HY7X4HHEAD11X1HX9X1HY7X4HHEAD
     111X1HX9X1HY7X4HHEAD/)
 1015 FORMAT(1H F12.2,2F10.2,F13.2,2F10.2,F13.2,2F10.2,F13.2,2F10.2)
 1016 FORMAT(1H158X14HTOO MANY ZONES)
 1017 FORMAT(1H 50X28HTOO MANY BOUNDARIES FOR ZONEI3)
 1018 FORMAT(1H 54X20HDATA INCORRECT, ZONEI3)
 1019 FORMAT(1H 54X23HSECTION TOO COMPLICATED)
 1020 FORMAT(1H 58X14HDATA INCORRECT)
 1021 FORMAT(1H )
 1022 FORMAT(1H'41X22HLARGEST RESIDUAL AFTERI4,15H ITERATIONS WASF8.3)
 1023 FORMAT(1H156X4HZONEI3,12H (CONTINUED)36X4HPAGEI4/)
 1024 FORMAT(1H 62X4HZONEI3/)
      END
```

Typical set of data

The first card of each set contains the number of zones, the tolerable residual, and the allowable number of iterations. The data for each zone follows next, and appears in batches of cards, each batch containing all the data for one zone. The first card of each batch contains the maximum desirable distance between nodes, the minimum and maximum permeabilities (in exponential form), and the direction of minimum permeability measured counter-clockwise from the horizontal. The remaining cards each contain the coordinates of the corner points of each zone in clockwise order; starting and finishing with the same point.

Following the zone data, one card appears for every constant potential face. Each card contains the coordinates of the points between which the face lies, i.e. X1, Y1, X2, Y2, followed by the head HA acting on the face. A dummy card consisting of zeros follows the constant potential face data.

The data relating to the impermeable boundaries follows next, each card containing the coordinates of the points between which the boundary lies. A dummy card consisting of zeros completes the set of data.

A further dummy card containing a zero follows the final set of data.

3	0.010	120		
5.00	6.00E-06	24.00E-06	90.00	
0.00	100.00			
50.00	100.00			
150.00	100.00			
200.00	100.00			
200.00	75.00			
0.00	75.00			
0.00	100.00			
5.00	8.00E-06	3.20E-05	90.00	
200.00	75.00			
200.00	50.00			
0.00	50.00			
0.00	75.00			
200.00	75.00			
5.00	6.40E-05	6.40E-05	0.00	
200.00	0.00			
0.00	0.00			
0.00	50.00			
200.00	50.00			
200.00	0.00			
0.00	100.00	50.00	100.00	100.00
150.00	100.00	200.00	100.00	0.00
0.00	0.00	0.00	0.00	0.00
200.00	100.00	200.00	75.00	
200.00	75.00	200.00	50.00	
200.00	50.00	200.00	0.00	
200.00	0.00	0.00	0.00	
0.00	0.00	0.00	50.00	
0.00	50.00	0.00	75.00	
0.00	75.00	0.00	100.00	
50.00	100.00	150.00	100.00	
0.00	0.00	0.00	0.00	

FIGURE D1. Typical set of data

A typical set of data is shown in FIGURE D1. This data relates to a flow region in which there are 3 zones. The minimum and maximum permeabilities of two zones are 6×10^{-6} and 24×10^{-6} cm/sec, and 8×10^{-6} and 32×10^{-6} cm/sec respectively and they are horizontally stratified, i.e., the direction of the minimum permeability is 90 degrees. The third zone is isotropic, and has a permeability value of 64×10^{-6} cm/sec. The tolerable residual is 0.01 and the maximum number of iterations permitted is 120.

Appendix E

TYPICAL SET OF OUTPUT

APPENDIX E

TYPICAL SET OF OUTPUT

As each set of data is analysed, the output is printed in tabular form. A typical example of output is presented on pages E3 to E7. The information therein is plotted in FIGURE **5.3(a)** in the text.

The first page of each set of output consists of the data describing the flow region, the number of iterations performed, and the largest residual in the final iteration.

Succeeding pages contain tabulations of the potential head values at every node in the flow region. There are four tables across each page, and each table consists of three columns. The numbers in the first and second columns are the x- and y- coordinates of a node; the potential head at the node is given in the third column.

ZONE NO	MINIMUM PERMEABILITY	MAXIMUM PERMEABILITY	DIRECTION OF MIN PERM
1	6.000E-06	2.400E-05	90.00
2	8.000E-06	3.200E-05	90.00
3	6.400E-05	ISOTROPIC	

LOCATIONS OF FACES OF CONSTANT POTENTIAL

X	Y	X	Y
0.	100.00	50.00	100.00
150.00	100.00	200.00	100.00

LOCATIONS OF IMPERMEABLE BOUNDARIES

X	Y	X	Y
200.00	100.00	200.00	75.00
200.00	75.00	200.00	50.00
200.00	50.00	200.00	0.
200.00	0.	0.	0.
0.	0.	0.	50.00
0.	50.00	0.	75.00
0.	75.00	0.	100.00
50.00	100.00	150.00	100.00

LARGEST RESIDUAL AFTER 120 ITERATIONS WAS 0.010

ZONE 1

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
150.00	100.00	0.	155.00	100.00	0.	160.00	100.00	0.	165.00	100.00	0.
170.00	100.00	0.	175.00	100.00	0.	180.00	100.00	0.	185.00	100.00	0.
190.00	100.00	0.	195.00	100.00	0.	200.00	100.00	0.			
155.00	97.50	4.97	160.00	97.50	4.47	165.00	97.50	4.12	170.00	97.50	3.87
175.00	97.50	3.69	180.00	97.50	3.55	185.00	97.50	3.46	190.00	97.50	3.39
195.00	97.50	3.36	200.00	97.50	3.34						
160.00	95.00	8.79	165.00	95.00	8.14	170.00	95.00	7.67	175.00	95.00	7.32
180.00	95.00	7.07	185.00	95.00	6.88	190.00	95.00	6.76	195.00	95.00	6.69
200.00	95.00	6.66									
165.00	92.50	11.99	170.00	92.50	11.34	175.00	92.50	10.86	180.00	92.50	10.50
185.00	92.50	10.24	190.00	92.50	10.07	195.00	92.50	9.97	200.00	92.50	9.93
170.00	90.00	14.85	175.00	90.00	14.27	180.00	90.00	13.83	185.00	90.00	13.52
190.00	90.00	13.30	195.00	90.00	13.17	200.00	90.00	13.13			
175.00	87.50	17.53	180.00	87.50	17.04	185.00	87.50	16.68	190.00	87.50	16.44
195.00	87.50	16.29	200.00	87.50	16.24						
180.00	85.00	20.12	185.00	85.00	19.73	190.00	85.00	19.47	195.00	85.00	19.31
200.00	85.00	19.26									
185.00	82.50	22.65	190.00	82.50	22.38	195.00	82.50	22.22	200.00	82.50	22.16
190.00	80.00	25.18	195.00	80.00	25.01	200.00	80.00	24.95			
195.00	77.50	27.69	200.00	77.50	27.63						
200.00	75.00	30.19									
50.00	100.00	100.00	60.00	100.00	83.89	70.00	100.00	72.80	80.00	100.00	64.26
90.00	100.00	56.99	100.00	100.00	50.26	110.00	100.00	43.50	120.00	100.00	36.19
130.00	100.00	27.55	140.00	100.00	16.32	150.00	100.00	0.			
65.00	97.50	76.84	75.00	97.50	67.76	85.00	97.50	60.22	95.00	97.50	53.43
105.00	97.50	46.85	115.00	97.50	40.04	125.00	97.50	32.55	135.00	97.50	23.98
145.00	97.50	14.23	155.00	97.50	4.97						
80.00	95.00	63.31	90.00	95.00	56.50	100.00	95.00	50.09	110.00	95.00	43.70
120.00	95.00	37.01	130.00	95.00	29.80	140.00	95.00	22.07	150.00	95.00	14.45
160.00	95.00	8.79									
95.00	92.50	53.12	105.00	92.50	47.07	115.00	92.50	40.96	125.00	92.50	34.61
135.00	92.50	28.02	145.00	92.50	21.47	155.00	92.50	15.72	165.00	92.50	11.99
110.00	90.00	44.46	120.00	90.00	38.73	130.00	90.00	32.91	140.00	90.00	27.13
150.00	90.00	21.77	160.00	90.00	17.47	170.00	90.00	14.85			
125.00	87.50	37.06	135.00	87.50	31.88	145.00	87.50	26.96	155.00	87.50	22.67
165.00	87.50	19.44	175.00	87.50	17.53						
140.00	85.00	31.44	150.00	85.00	27.35	160.00	85.00	23.97	170.00	85.00	21.53
180.00	85.00	20.11									
155.00	82.50	28.18	165.00	82.50	25.53	175.00	82.50	23.70	185.00	82.50	22.65
170.00	80.00	27.28	180.00	80.00	25.92	190.00	80.00	25.18			
185.00	77.50	28.18	195.00	77.50	27.69						
200.00	75.00	30.19									
50.00	100.00	100.00	65.00	97.50	76.84	80.00	95.00	63.31	95.00	92.50	53.12
110.00	90.00	44.46	125.00	87.50	37.06	140.00	85.00	31.44	155.00	82.50	28.18
170.00	80.00	27.28	185.00	77.50	28.18	200.00	75.00	30.19			
45.00	97.50	95.10	60.00	95.00	78.89	75.00	92.50	65.99	90.00	90.00	55.92
105.00	87.50	47.56	120.00	85.00	40.59	135.00	82.50	35.25	150.00	80.00	31.94
165.00	77.50	30.71	180.00	75.00	31.19						
40.00	95.00	91.34	55.00	92.50	79.31	70.00	90.00	67.80	85.00	87.50	58.21
100.00	85.00	50.23	115.00	82.50	43.65	130.00	80.00	38.60	145.00	77.50	35.34
160.00	75.00	33.93									

ZONE 1 (CONTINUED)

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
35.00	92.50	88.18	50.00	90.00	78.81	65.00	87.50	68.83	80.00	85.00	59.96
95.00	82.50	52.45	110.00	80.00	46.27	125.00	77.50	41.52	140.00	75.00	38.37
30.00	90.00	85.35	45.00	87.50	77.78	60.00	85.00	69.20	75.00	82.50	61.17
90.00	80.00	54.21	105.00	77.50	48.46	120.00	75.00	44.02			
25.00	87.50	82.71	40.00	85.00	76.41	55.00	82.50	69.06	70.00	80.00	61.89
85.00	77.50	55.53	100.00	75.00	50.22						
20.00	85.00	80.15	35.00	82.50	74.81	50.00	80.00	68.50	65.00	77.50	62.16
80.00	75.00	56.41									
15.00	82.50	77.64	30.00	80.00	73.05	45.00	77.50	67.61	60.00	75.00	62.04
10.00	80.00	75.14	25.00	77.50	71.16	40.00	75.00	66.46			
5.00	77.50	72.64	20.00	75.00	69.18						
-0.	75.00	70.17									
-0.	100.00	100.00	5.00	100.00	100.00	10.00	100.00	100.00	15.00	100.00	100.00
20.00	100.00	100.00	25.00	100.00	100.00	30.00	100.00	100.00	35.00	100.00	100.00
40.00	100.00	100.00	45.00	100.00	100.00	50.00	100.00	100.00			
-0.	97.50	96.70	5.00	97.50	96.69	10.00	97.50	96.65	15.00	97.50	96.59
20.00	97.50	96.50	25.00	97.50	96.37	30.00	97.50	96.19	35.00	97.50	95.94
40.00	97.50	95.59	45.00	97.50	95.10						
-0.	95.00	93.43	5.00	95.00	93.41	10.00	95.00	93.34	15.00	95.00	93.22
20.00	95.00	93.04	25.00	95.00	92.78	30.00	95.00	92.44	35.00	95.00	91.97
40.00	95.00	91.34									
-0.	92.50	90.20	5.00	92.50	90.17	10.00	92.50	90.07	15.00	92.50	89.90
20.00	92.50	89.65	25.00	92.50	89.29	30.00	92.50	88.81	35.00	92.50	88.18
-0.	90.00	87.04	5.00	90.00	87.00	10.00	90.00	86.88	15.00	90.00	86.67
20.00	90.00	86.36	25.00	90.00	85.93	30.00	90.00	85.35			
-0.	87.50	83.97	5.00	87.50	83.92	10.00	87.50	83.78	15.00	87.50	83.54
20.00	87.50	83.19	25.00	87.50	82.71						
-0.	85.00	80.99	5.00	85.00	80.94	10.00	85.00	80.79	15.00	85.00	80.53
20.00	85.00	80.15									
-0.	82.50	78.11	5.00	82.50	78.06	10.00	82.50	77.91	15.00	82.50	77.64
-0.	80.00	75.35	5.00	80.00	75.30	10.00	80.00	75.14			
-0.	77.50	72.70	5.00	77.50	72.65						
-0.	75.00	70.17									

ZONE 2

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
200.00	50.00	45.26	180.00	50.00	45.51	160.00	50.00	46.20	140.00	50.00	47.29
120.00	50.00	48.67	100.00	50.00	50.19	80.00	50.00	51.71	60.00	50.00	53.08
40.00	50.00	54.17	20.00	50.00	54.86	0.	50.00	55.09			
180.00	52.50	44.35	160.00	52.50	45.21	140.00	52.50	46.58	120.00	52.50	48.29
100.00	52.50	50.20	80.00	52.50	52.10	60.00	52.50	53.81	40.00	52.50	55.15
20.00	52.50	56.02	0.	52.50	56.32						
160.00	55.00	44.19	140.00	55.00	45.83	120.00	55.00	47.91	100.00	55.00	50.20
80.00	55.00	52.49	60.00	55.00	54.56	40.00	55.00	56.19	20.00	55.00	57.23
0.	55.00	57.59									
140.00	57.50	45.06	120.00	57.50	47.50	100.00	57.50	50.20	80.00	57.50	52.90
60.00	57.50	55.34	40.00	57.50	57.26	20.00	57.50	58.48	0.	57.50	58.90
120.00	60.00	47.08	100.00	60.00	50.21	80.00	60.00	53.33	60.00	60.00	56.15
40.00	60.00	58.37	20.00	60.00	59.78	0.	60.00	60.27			

ZONE 2 (CONTINUED)

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
100.00	62.50	50.21	80.00	62.50	53.78	60.00	62.50	57.00	40.00	62.50	59.54
20.00	62.50	61.15	0.	62.50	61.71						
80.00	65.00	54.25	60.00	65.00	57.90	40.00	65.00	60.77	20.00	65.00	62.58
0.	65.00	63.22									
60.00	67.50	58.84	40.00	67.50	62.07	20.00	67.50	64.10	0.	67.50	64.81
40.00	70.00	63.44	20.00	70.00	65.69	0.	70.00	66.49			
20.00	72.50	67.38	0.	72.50	68.27						
-0.	75.00	70.17									
200.00	75.00	30.19	200.00	72.50	32.10	200.00	70.00	33.89	200.00	67.50	35.58
200.00	65.00	37.17	200.00	62.50	38.68	200.00	60.00	40.12	200.00	57.50	41.48
200.00	55.00	42.79	200.00	52.50	44.05	200.00	50.00	45.26			
180.00	75.00	31.19	180.00	72.50	32.99	180.00	70.00	34.69	180.00	67.50	36.29
180.00	65.00	37.80	180.00	62.50	39.24	180.00	60.00	40.60	180.00	57.50	41.90
180.00	55.00	43.14	180.00	52.50	44.34						
160.00	75.00	33.93	160.00	72.50	35.49	160.00	70.00	36.95	160.00	67.50	38.33
160.00	65.00	39.63	160.00	62.50	40.85	160.00	60.00	42.02	160.00	57.50	43.13
160.00	55.00	44.19									
140.00	75.00	38.37	140.00	72.50	39.51	140.00	70.00	40.57	140.00	67.50	41.57
140.00	65.00	42.51	140.00	62.50	43.40	140.00	60.00	44.25	140.00	57.50	45.05
120.00	75.00	44.03	120.00	72.50	44.61	120.00	70.00	45.16	120.00	67.50	45.68
120.00	65.00	46.17	120.00	62.50	46.63	120.00	60.00	47.08			
100.00	75.00	50.22	100.00	72.50	50.22	100.00	70.00	50.22	100.00	67.50	50.22
100.00	65.00	50.21	100.00	62.50	50.21						
80.00	75.00	56.41	80.00	72.50	55.82	80.00	70.00	55.27	80.00	67.50	54.75
80.00	65.00	54.25									
60.00	75.00	62.04	60.00	72.50	60.91	60.00	70.00	59.84	60.00	67.50	58.84
40.00	75.00	66.46	40.00	72.50	64.90	40.00	70.00	63.44			
20.00	75.00	69.18	20.00	72.50	67.39						
-0.	75.00	70.17									

ZONE 3

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
0.	0.	53.61	0.	5.00	53.63	0.	10.00	53.68	0.	15.00	53.75
0.	20.00	53.85	0.	25.00	53.99	0.	30.00	54.15	0.	35.00	54.34
0.	40.00	54.56	0.	45.00	54.82	0.	50.00	55.11			
20.00	5.00	53.46	20.00	10.00	53.51	20.00	15.00	53.58	20.00	20.00	53.68
20.00	25.00	53.81	20.00	30.00	53.96	20.00	35.00	54.14	20.00	40.00	54.35
20.00	45.00	54.60	20.00	50.00	54.87						
40.00	10.00	53.02	40.00	15.00	53.08	40.00	20.00	53.16	40.00	25.00	53.27
40.00	30.00	53.40	40.00	35.00	53.56	40.00	40.00	53.74	40.00	45.00	53.94
40.00	50.00	54.18									
60.00	15.00	52.29	60.00	20.00	52.35	60.00	25.00	52.43	60.00	30.00	52.53
60.00	35.00	52.64	60.00	40.00	52.77	60.00	45.00	52.92	60.00	50.00	53.09
80.00	20.00	51.32	80.00	25.00	51.37	80.00	30.00	51.42	80.00	35.00	51.48
80.00	40.00	51.55	80.00	45.00	51.63	80.00	50.00	51.72			
100.00	25.00	50.19	100.00	30.00	50.19	100.00	35.00	50.19	100.00	40.00	50.19
100.00	45.00	50.19	100.00	50.00	50.19						
120.00	30.00	48.95	120.00	35.00	48.90	120.00	40.00	48.83	120.00	45.00	48.75
120.00	50.00	48.66									

ZONE 3 (CONTINUED)

X	Y	HEAD	X	Y	HEAD	X	Y	HEAD	X	Y	HEAD
140.00	35.00	47.73	140.00	40.00	47.60	140.00	45.00	47.45	140.00	50.00	47.29
160.00	40.00	46.63	160.00	45.00	46.43	160.00	50.00	46.19			
180.00	45.00	45.77	180.00	50.00	45.50						
200.00	50.00	45.26									
200.00	0.	46.73	180.00	0.	46.90	160.00	0.	47.38	140.00	0.	48.15
120.00	0.	49.11	100.00	0.	50.18	80.00	0.	51.25	60.00	0.	52.21
40.00	0.	52.97	20.00	0.	53.46	0.	0.	53.62			
200.00	5.00	46.71	180.00	5.00	46.88	160.00	5.00	47.37	140.00	5.00	48.14
120.00	5.00	49.11	100.00	5.00	50.18	80.00	5.00	51.26	60.00	5.00	52.22
40.00	5.00	52.99	20.00	5.00	53.48						
200.00	10.00	46.67	180.00	10.00	46.83	160.00	10.00	47.33	140.00	10.00	48.11
120.00	10.00	49.09	100.00	10.00	50.18	80.00	10.00	51.27	60.00	10.00	52.25
40.00	10.00	53.03									
200.00	15.00	46.59	180.00	15.00	46.76	160.00	15.00	47.27	140.00	15.00	48.06
120.00	15.00	49.07	100.00	15.00	50.18	80.00	15.00	51.30	60.00	15.00	52.30
200.00	20.00	46.49	180.00	20.00	46.66	160.00	20.00	47.19	140.00	20.00	48.00
120.00	20.00	49.04	100.00	20.00	50.18	80.00	20.00	51.33			
200.00	25.00	46.36	180.00	25.00	46.54	160.00	25.00	47.08	140.00	25.00	47.93
120.00	25.00	49.00	100.00	25.00	50.19						
200.00	30.00	46.20	180.00	30.00	46.39	160.00	30.00	46.95	140.00	30.00	47.84
120.00	30.00	48.95									
200.00	35.00	46.01	180.00	35.00	46.21	160.00	35.00	46.80	140.00	35.00	47.72
200.00	40.00	45.79	180.00	40.00	46.00	160.00	40.00	46.62			
200.00	45.00	45.54	180.00	45.00	45.76						
200.00	50.00	45.25									

Appendix F

DERIVATION OF THE FINITE DIFFERENCE EQUATION FOR THE POINT AT AN APEX OF SEVERAL TRIANGLES

APPENDIX F

DERIVATION OF THE FINITE DIFFERENCE EQUATION FOR THE POINT AT AN APEX OF SEVERAL TRIANGLES

A derivation of equation (3.43) in the text was obtained after this thesis was written. This derivation is presented below.

Consider a polygon ABCDE enclosing the apex O of several triangles, as shown in FIGURE F1, where the Ou and Ov axes of each triangle form the boundaries and the Ow axes are not shown. The points U_1', U_2', U_3', U_4', U_5' are nodes adjacent to the point O along the Ou axes of triangles 1, 2, 3, 4, 5 respectively. A, B, C, D, E are situated at the mid-points of OU_1', OU_2', OU_3', OU_4', OU_5' respectively so that the linear dimensions of the polygon ABCDE are one-half of those of the polygon $U_1'U_2'U_3'U_4'U_5'$.

As in SECTION 3.4, use is made of the concept that any distance along the principal permeability directions Ox_1 and Oy_1, and also the direction Ov_1, in triangle 1 can be regarded as a function of distances along the Ou_1 and Ow_1 axes to give

$$(h)_{x1} = (h)_{u1}.(u_1)_{x1} + (h)_{w1}.(w_1)_{x1} \, , \tag{F1a}$$

$$(h)_{y1} = (h)_{u1}.(u_1)_{y1} + (h)_{w1}.(w_1)_{y1} \, , \tag{F1b}$$

and

$$(h)_{v1} = (h)_{u1}.(u_1)_{v1} + (h)_{w1}.(w_1)_{v1} \, . \tag{F1c}$$

Substituting expressions for $(u_1)_{x1}$, $(w_1)_{x1}$, $(u_1)_{y1}$, $(w_1)_{y1}$, $(u_1)_{v1}$, $(w_1)_{v1}$ obtained from equations (3.4c to h) in equations (F1a, b, c),

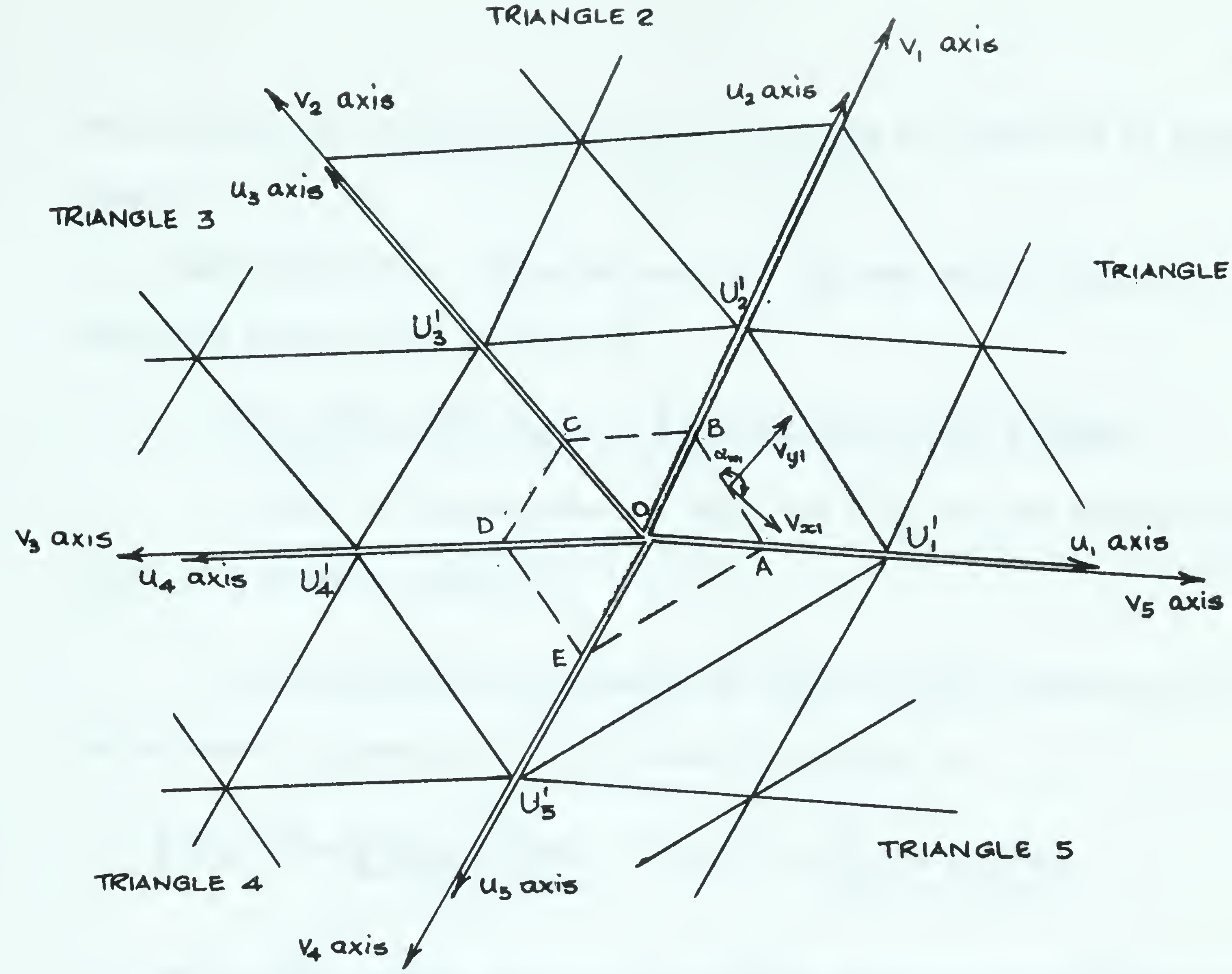

FIGURE F1. The apex of several triangles.

$$(h)_{x1} = (h)_{u1} \cdot \frac{\sin \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} - (h)_{w1} \cdot \frac{\sin \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})}, \tag{F2a}$$

$$(h)_{y1} = -(h)_{u1} \cdot \frac{\cos \alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} + (h)_{w1} \cdot \frac{\cos \alpha_{u1}}{\sin(\alpha_{w1} - \alpha_{u1})}, \tag{F2b}$$

$$(h)_{v1} = (h)_{u1} \cdot \frac{\sin(\alpha_{w1} - \alpha_{v1})}{\sin(\alpha_{w1} - \alpha_{u1})} - (h)_{w1} \cdot \frac{\sin(\alpha_{u1} - \alpha_{v1})}{\sin(\alpha_{w1} - \alpha_{u1})}. \tag{F2c}$$

The term $(h)_{w1}$ is eliminated from equations (F2a, b, c), giving

$$(h)_{x1} = (h)_{u1} \frac{\sin \alpha_{v1}}{\sin(\alpha_{v1} - \alpha_{u1})} - (h)_{v1} \frac{\sin \alpha_{u1}}{\sin(\alpha_{v1} - \alpha_{u1})}, \tag{F3a}$$

$$(h)_{y1} = -(h)_{u1} \frac{\cos \alpha_{v1}}{\sin(\alpha_{v1} - \alpha_{u1})} + (h)_{v1} \frac{\cos \alpha_{u1}}{\sin(\alpha_{v1} - \alpha_{u1})}. \tag{F3b}$$

Assuming that the velocity components v_{x1} and v_{y1} in the principal permeability directions are uniform along the length AB in FIGURE F1, the flow across AB is found by resolving the components v_{x1} and v_{y1} in a direction

perpendicular to AB and multiplying the result by the length AB ($= \frac{1}{2}.\Delta_1 w_1$); that is,

$$\text{flow across AB} = (v_{x1} \sin \alpha_{w1} - v_{y1} \cos \alpha_{w1}).\tfrac{1}{2}.\Delta_1 w_1,$$

which, by Darcy's law, is equal to

$$-(k_{x1}\,(h)_{x1} \sin \alpha_{w1} - k_{y1}\,(h)_{y1} \cos \alpha_{w1}).\tfrac{1}{2}.\Delta_1 w_1,$$

where, in this expression, $(h)_{x1}$ and $(h)_{y1}$ are the partial derivatives of h at points along AB.

Substituting the expansions for $(h)_{x1}$ and $(h)_{y1}$, (equations F3a, b), in the above expression, the flow across AB is equal to

$$-\left\{k_{x1}(h)_{u1}\frac{\sin\alpha_{v1}\sin\alpha_{w1}}{\sin(\alpha_{v1}-\alpha_{u1})} - k_{x1}(h)_{v1}\frac{\sin\alpha_{u1}\sin\alpha_{w1}}{\sin(\alpha_{v1}-\alpha_{u1})}\right.$$

$$\left. + k_{y1}(h)_{u1}\frac{\cos\alpha_{v1}\cos\alpha_{w1}}{\sin(\alpha_{v1}-\alpha_{u1})} - k_{y1}(h)_{v1}\frac{\cos\alpha_{u1}\cos\alpha_{w1}}{\sin(\alpha_{v1}-\alpha_{u1})}\right\} \times \frac{(\Delta_1 w_1)}{2}, \quad \text{(F4)}$$

where, in this expression, $(h)_{u1}$ and $(h)_{v1}$ are the partial derivatives of h at points along AB. These terms are given approximately by the finite difference expressions

$$(h)_{u1} = \frac{h'_{u1} - h_o}{\Delta_1 u_1}, \quad \text{(F5a)}, \qquad (h)_{v1} = \frac{h'_{v1} - h_o}{\Delta_1 v_1}, \quad \text{(F5b)}$$

By the sine rule on triangle $OU'_1U'_2$,

$$\frac{\Delta_1 u_1}{\sin(\alpha_{w1}-\alpha_{v1})} = \frac{\Delta_1 v_1}{\sin(\alpha_{w1}-\alpha_{u1})} = \frac{\Delta_1 w_1}{\sin(\alpha_{v1}-\alpha_{u1})}.$$

Using these relationships and the fact that $h'_{v1} = h'_{u2}$, equations (F5a,b) are altered to

$$(h)_{u1} = \frac{h'_{u1} - h_o}{\Delta_1 w_1}.\frac{\sin(\alpha_{v1}-\alpha_{u1})}{\sin(\alpha_{w1}-\alpha_{v1})}, \quad \text{(F6a)}$$

and $$(h)_{v1} = \frac{h'_{u2} - h_o}{\Delta_1 w_1} \cdot \frac{\sin(\alpha_{v1} - \alpha_{u1})}{\sin(\alpha_{w1} - \alpha_{u1})} \, . \qquad \text{(F6b)}$$

Substituting equations (F6a,b) in equation (F4), and rearranging,

Flow across AB =

$$\left\{ - \frac{h'_{u1} - h_o}{\Delta_1 w_1} \cdot \frac{k_{x1} \sin\alpha_{v1} \sin\alpha_{w1} + k_{y1} \cos\alpha_{v1} \cos\alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{v1})} + \frac{h'_{u2} - h_o}{\Delta_1 w_1} \cdot \frac{k_{x1} \sin\alpha_{u1} \sin\alpha_{w1} + k_{y1} \cos\alpha_{u1} \cos\alpha_{w1}}{\sin(\alpha_{w1} - \alpha_{u1})} \right\} \times \frac{(\Delta_1 w_1)}{2}$$

$$= \frac{C_{u1} h'_{u1} + C_{v1} h'_{u2} - (C_{u1} + C_{v1}) h_o}{2} \qquad \text{(F7)}$$

where C_{u1}, C_{v1} are coefficients appertaining to zone 1 having the same expansions as equations (3.19a, b).

Continuity requires that the net flow across sides of the polygon ABCDE be zero; hence, by summing expressions similar to equation (F7),

$$\frac{C_{u1}+C_{v5}}{2} h'_{u1} + \frac{C_{u2}+C_{v1}}{2} h'_{u2} + \frac{C_{u3}+C_{v2}}{2} h'_{u3} + \frac{C_{u4}+C_{v3}}{2} h'_{u4} + \frac{C_{u5}+C_{v4}}{2} h'_{u5} - \left\{ \frac{C_{u1}+C_{u2}+C_{u3}+C_{u4}+C_{u5}+C_{v1}+C_{v2}+C_{v3}+C_{v4}+C_{v5}}{2} \right\} h_o = 0.$$

This equation can be stated in the form of equation (3.43) for cases in which there are n triangles with apices at O.

$$\left. \begin{aligned} &(C_{u1}+C_{vn}) h'_{u1} + (C_{u2}+C_{v1}) h'_{u2} + \cdots + (C_{un} + C_{vn-1}) h'_{un} \\ &\quad - (C_{u1}+C_{v1}+C_{u2}+C_{v2}+ \cdots + C_{un}+C_{vn}) h_o = 0. \end{aligned} \right\} \qquad (3.43)$$